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Improved confidence intervals for the variance and standard deviation of a normal population.

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IMPROVED CONFIDENCE INTERVALS FOR THE VARIANCE AND STANDARD DEVIATION OF A NORMAL POPULATION

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THESIS

IMPROVED CONFIDENCE INTERVALS FOR THE VARIANCE AND STANDARD DEVIATION OF A NORMAL POPULATION

by

Archie Andrew Turner, III

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Improved Confidence Intervals for the
Variance and Standard Deviation of a
Normal Population

by

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ABSTRACT

Shortest confidence intervals for the variance of a normal population are found as superior alternatives to the widely used equal-tail confidence intervals. Extensive tables of high precision are presented which enable the user to construct easily minimum-length confidence intervals for the variance, the standard deviation, and shortest unbiased confidence intervals. Characteristics and improved performance of the alternative confidence intervals are discussed in detail and illustrated graphically with an emphasis on the optimal distribution of complementary tail area between the upper and lower tails.

TABLE OF CONTENTS

I. INTRODUCTION

A. PURPOSE

The purpose of this thesis is to describe improved confidence intervals for the variance and the standard deviation of a random sample drawn from a normal population. Consideration is limited to those classes of confidence intervals which are based on the sum of the squared deviations of the observations from the mean.

The question of optimal confidence intervals for the variance of a normal population has been treated earlier by Tate and Klett [7]. The contents of this thesis comprise a considerable extension of their work.

B. BACKGROUND

The most widely used method of constructing confidence intervals for the variance or standard deviation of a normal population involves the practice of evenly splitting the tails of the appropriate chi-square distribution. The popularity which this procedure has enjoyed over the years is due primarily to the fact that, until recently, the only inexpensive way to construct such intervals depended upon the use of published tables of the chi-square distribution. When the degrees of freedom are large, the confidence intervals which result from the equal-tails procedure provide satisfactory results. This is a consequence of the convergence

of the chi-square and normal probability distributions for large degrees of freedom. When the degrees of freedom are small however, use of the equal-tail procedure results in confidence intervals which are unsatisfactory in all respects save computational ease.

Fortunately, the difficulties involved with the construction of improved confidence intervals for the variance or standard deviation are of a computational rather than conceptual nature. Thus, in light of the computational power of present day computer hardware, use of the equal-tail confidence intervals may be phased out for virtually all practitioners.

This thesis describes two classes of confidence intervals which offer significant improvements over equivalent equal-tail confidence intervals. The benefits which accrue from the use of these improved confidence intervals are discussed in detail, and illustrated graphically.

C. IMPROVED CONFIDENCE INTERVALS

Although there is no universal agreement as to precisely what constitutes an "optimal" confidence interval, for the purposes of this thesis two attributes, interval length and unbiasedness, were chosen as measures of effectiveness. It is important to observe that these criteria, although intuitively pleasing, are arbitrary and neither exhaust nor dominate a list of possible alternative concepts of optimality. In general however, it seems eminently reasonable that all other

factors being equal, a shorter confidence interval is to be preferred to a longer interval, and that an unbiased confidence interval, in the sense of Neyman [2], is to be preferred to an interval which exhibits bias.

This thesis then will be concerned with constructing confidence intervals for the variance and standard deviation which are optimal in the sense that they are either of minimum length, or the shortest of the unbiased confidence intervals.

D. CONTENTS OF THESIS

Section II of the thesis contains a comprehensive discussion of the logic and methodology employed in determining the values from the chi-square distribution which result in the shortest unbiased, or the minimum-length confidence intervals. A discussion concerning the accuracy of the method employed, and its effect on the accuracy of the tabled values appears in Section III. Conclusions concerning the characteristics and performance of the alternative intervals of interest are presented in Section IV together with some suggested "thumb rules" for users.

Several appendices follow. Appendix A contains mathematical derivations of the analytical relations used in the optimization process. Appendix B contains the tabled values which may be used in constructing the improved/optimal confidence intervals discussed in the thesis. Several illustrations pertinent to the problem and the results of the thesis follow in Appendix C.

II. DISCUSSION OF LOGIC AND METHODOLOGY

A. MINIMUM-LENGTH CONFIDENCE INTERVALS FOR THE VARIANCE AND STANDARD DEVIATION

Assume that $\{X_1, X_2, X_3, \ldots, X_N\}$ is a random sample from a normal population of unknown variance.

Let γ, $(0 \leq \gamma \leq 1)$ be an arbitrary level of confidence. Define

$$S^2 = \sum_{1}^{N} (X_i - \overline{X})^2 ,$$

where

$$\overline{X} = \frac{\sum_{1}^{N} X_i}{N}$$

is the sample mean, and N is the number of observations.

Under the assumption that each X_i is an independent, normally distributed random variable, it is easily shown that the statistic

$$\frac{S^2}{\sigma^2}$$

is a chi-square random variable with (N-1) degrees of freedom.[1]

---

[1] The degrees of freedom are N if the value of the population mean is known and used in place of the sample mean when computing S^2.

In order to construct the desired confidence interval based upon the preceeding assumptions, two values C1 and C2 are selected arbitrarily but subject to the constraint

$$\int_{C1}^{C2} g_K(x)\,dx = \gamma$$

where $g_K(x)$ denotes the chi-square probability density function with K degrees of freedom. As a matter of convenience, C1 and C2 are usually chosen such that

$$\int_{0}^{C1} g_K(x)\,dx = \int_{C2}^{\infty} g_K(x)\,dx = \frac{1-\gamma}{2},$$

that is, to distribute the complementary tail area evenly between the upper and lower tails, or "split the tails". Because $\frac{S^2}{\sigma^2}$ is a chi-square random variable, direct substitution results in the desired probability statement, $P(C1 \leq \frac{S^2}{\sigma^2} \leq C2) = \gamma$. Simple algebra is all that remains in forming the desired confidence interval. Thus $P(\frac{S^2}{C2} \leq \sigma^2 \leq \frac{S^2}{C1}) = \gamma$, and the interval $(\frac{S^2}{C2}, \frac{S^2}{C1})$ is a gamma level confidence interval for the variance, of length $(\frac{S^2}{C1} - \frac{S^2}{C2})$. Hence, construction of the minimum-length confidence interval for the variance is simply a matter of finding the pair (C1,C2), which minimizes the length expression $(\frac{S^2}{C1} - \frac{S^2}{C2})$, while simultaneously satisfying the probability integral constraint. In the context of non-linear programming this problem may be stated as the constrained nonlinear optimization:

Minimize: $S^2(\frac{1}{C1} - \frac{1}{C2})$

$$\text{Subject to:} \quad \int_0^{C1} g_K(x)\,dx + \int_{C2}^{\infty} g_K(x)\,dx = 1 - \gamma.$$

In finding the optimal solution to this problem, there is but a single degree of freedom, for once the value of either C1 or C2 is chosen the pair (C1,C2) is determined by the feasibility requirements imposed by the constraint relation. Hence the problem may be treated as a single variable optimization.

Arbitrarily, designate C1 as the independent decision variable. Clearly, the value of C1 is limited to the interval $(0,G_K^{-1}(1-\gamma))$ where $G_K^{-1}(1-\gamma)$ represents the inverse cumulative distribution function of the appropriate chi-square distribution evaluated at $(1-\gamma)$. As C1 varies from zero to $G_K^{-1}(1-\gamma)$ the constraint relation forces C2 to assume values which range from $G_K^{-1}(\gamma)$ to infinity. Thus it is easily seen that the problem of constructing the minimum-length confidence interval for the variance is merely a single variable line search over the interval $(0,G_K^{-1}(1-\gamma))$ for the unique value of C1 which implies the pair (C1,C2) that minimizes the length expression $S^2(\frac{1}{C1} - \frac{1}{C2})$. The statistic S^2, being a function only of the data, is not involved in the optimization process.

The problem of constructing the minimum-length confidence interval for the standard deviation under the same assumptions may be formulated and solved in a completely analogous manner. It is necessary only to observe that in this case the procedure results in the interval

$$\left(\frac{S}{\sqrt{C2}}, \frac{S}{\sqrt{C1}}\right)$$

of length

$$\left(\frac{S}{\sqrt{C1}} - \frac{S}{\sqrt{C2}}\right).$$

This problem then may also be formulated as a constrained nonlinear optimization problem:

$$\text{Minimize:} \quad S\left(\frac{1}{\sqrt{C1}} - \frac{1}{\sqrt{C2}}\right)$$

$$\text{Subject to:} \quad \int_0^{C1} g_K(x)\,dx + \int_{C2}^{\infty} g_K(x)\,dx = 1 - \gamma$$

The rationale involved in the solution of this problem is identical to that which was applied to the problem of the variance, and is therefore omitted.

B. THE SHORTEST UNBIASED CONFIDENCE INTERVAL

A confidence interval is said to be unbiased, in the sense of Neyman [2], if it satisfies the following criteria,

$P(\ell\ell \leq \theta \leq u\ell | \theta=\theta_o) = \gamma$ and $P(\ell\ell \leq \theta \leq u\ell | \theta\neq\theta_o) \leq \gamma$ where $\ell\ell$ and $u\ell$ are respectively the lower and upper endpoints of the confidence interval, θ the parameter of interest, and θ_o the true value of the parameter of interest. Thus, an unbiased confidence interval is constructed in such a manner as to ensure that there is no value more likely to lie within the resulting interval than the true value of the parameter. Specifically, it is required that the true value of the parameter be contained with probability γ, but that any other incorrect value be contained with probability less than or equal to γ.

An alternative, more intuitive, explanation of the unbiased confidence interval is possible due to the equivalence of the concepts of confidence intervals and hypothesis testing regions of acceptance.

Consider a test of the hypotheses:

$$H_o: \sigma^2 = \sigma_o^2$$

$$H_1: \sigma^2 \neq \sigma_o^2$$

As before, the statistic $\frac{S^2}{\sigma^2}$ is easily shown to be a chi-square random variable. Consider the test statistic (TS) $\frac{S^2}{\sigma_o^2}$. If the null hypothesis is true, the test statistic is a chi-square random variable. In general however, TS = $\frac{S^2}{\sigma_o^2}$ is the product of a constant, $(\frac{\sigma^2}{\sigma_o^2})$ and a chi-square random variable $\frac{S^2}{\sigma^2}$.

In order to construct the desired region of acceptance for the test, it is necessary to select values C1 and C2 such that

$$\int_{C1}^{C2} g_K(x)\,dx = 1 - \alpha$$

where α is the pre-selected level of significance, or size, of the test. In theory, any two values of C1 and C2 which satisfy the constraint relation may be chosen, as before however common practice is to choose C1 and C2 such that

$$\int_{0}^{C1} g_K(x)\,dx = \int_{C2}^{\infty} g_K(x)\,dx = \frac{1-\alpha}{2},$$

or to "split the tails".

If the test statistic assumes a value which lies within the interval (C1,C2) the null hypothesis is not rejected. Otherwise the null hypothesis is rejected in favor of the alternative.

In conducting such a test, it seems reasonable that one would prefer a region of acceptance so constructed that the probability of rejecting the null hypothesis is at a minimum when it is in fact correct. Also, at this minimum the probability of rejection should equal α, the level of significance. A region which satisfies these seemingly reasonable requirements is said to be unbiased. Neither the equal-tail, nor

the minimum-length regions of acceptance (or confidence intervals) are unbiased. Reference to figure 1 which illustrates the power functions of these three regions of acceptance vividly illustrates the problem of bias in regions of acceptance (and equivalently, confidence intervals).

In order to construct the unbiased region of acceptance it is necessary to choose values C1 and C2 such that

$$
\begin{aligned}
P(\text{REJECT } H_o) &= 1 - P(C1 \le TS \le C2) \\
&= 1 - P\left(C1 \le \frac{\sigma^2}{\sigma_o^2} \chi_K^2 \le C2\right) \\
&= 1 - P\left(\frac{\sigma_o^2}{\sigma^2} C1 \le \chi_K^2 \le \frac{\sigma_o^2}{\sigma^2} C2\right) \\
&= 1 - \int_{\frac{\sigma_o^2}{\sigma^2}}^{\frac{\sigma_o^2}{\sigma^2}} g_K(x)\,dx
\end{aligned}
$$

is at a minimum, and equal to α when the null hypothesis is in fact correct ($\frac{\sigma_o^2}{\sigma^2} = 1$).

Equivalently, it must be that

$$\int_{C1}^{C2} g_K(x)\,dx = 1 - \alpha$$

and

$$\left.\frac{d[P(\text{REJECT } H_0)]}{d\left(\frac{\sigma_0^2}{\sigma^2}\right)}\right|_{\left(\frac{\sigma_0^2}{\sigma^2}\right)=1} = 0$$

which in turn implies that $C1g_K(C1) - C2g_k(C2) = 0$. Of course, these conditions are identical to those established by Neyman [2] for constructing the shortest unbiased confidence interval.

In passing, attention is drawn to a curious fact. There exists an analogy of this problem to the previous two problems involving minimum length. Suppose one asks the question "What function of the variance produces minimum-length confidence intervals whose (C1,C2) values satisfy the above (unbiased) equation?" The answer is easily shown to be the logarithm of the variance. This result is surely related to a similar result of Scheffé Ref. [5] for confidence intervals for the ratio of two variances using the F distribution.

An alternative method of determining the values C1 and C2 which result in the shortest unbiased interval was found in the work of Tate and Klett [7] and involves finding the pair (C1,C2) which minimizes the ratio of C2 over C1 subject to the constraint

$$\int_{C1}^{C2} g_K(x)dx = 1 - \alpha .$$

Of course this is equivalent to the "logarithmic transform" mentioned in the previous paragraph.

Although this method lacks intuitive motivation it is easily shown to be equivalent (see Appendix A) and because it permits a solution similar to that of the minimum-length intervals was used in this thesis. Thus, in order to find the values (C1,C2) which result in the shortest unbiased region of acceptance/confidence interval, it is necessary to solve the constrained, nonlinear program:

$$\text{Minimize:} \quad \left(\frac{C2}{C1}\right)$$

$$\text{Subject to:} \quad \int_{C1}^{C2} g_K(x)\,dx = (1-\alpha)(\gamma)$$

The logic used to determine the solution to this optimization problem is identical to that applied earlier in the case of the minimum-length intervals and is therefore omitted.

C. SUMMARY

Three problems are addressed:

(1) The shortest confidence interval for the variance.

(2) The shortest confidence interval for the standard deviation.

(3) The shortest unbiased confidence interval.

Each of the three problems may be expressed as a constrained nonlinear optimization.

To find the pair (C1,C2) which results in the shortest confidence interval for the variance it is necessary to solve the nonlinear optimization:

$$\text{Minimize:} \quad \left(\frac{1}{C1} - \frac{1}{C2}\right)$$

$$\text{Subject to:} \quad \int_0^{C1} g_K(x)\,dx + \int_{C2}^{\infty} g_K(x)\,dx = 1 - \gamma$$

To find the pair (C1,C2) which results in the minimum-length confidence interval for the standard deviation it is necessary to solve the constrained nonlinear optimization:

$$\text{Minimize:} \quad \left(\frac{1}{\sqrt{C1}} - \frac{1}{\sqrt{C2}}\right)$$

$$\text{Subject to:} \quad \int_0^{C1} g_K(x)\,dx + \int_{C2}^{\infty} g_K(x)\,dx = 1 - \gamma$$

To determine the pair (C1,C2) which results in Neyman's shortest unbiased confidence interval it is necessary to solve the constrained nonlinear optimization:

$$\text{Minimize:} \quad \left(\frac{C2}{C1}\right)$$

$$\text{Subject to:} \quad \int_0^{C1} g_K(x)\,dx + \int_{C2}^{\infty} g_K(x)\,dx = 1 - \gamma.$$

Each of the three problems may be solved by means of a single variable line search of the interval $(0, G_K(1-\gamma))$ for the unique value of C1 which results in the feasible pair (C1,C2) which minimizes the appropriate objective function.

III. DISCUSSION OF ACCURACY

The values which appear in the tables of Appendix B are accurate to the six decimal places presented. This conclusion is based upon the precision with which the optimal value of the decision variable C1 was determined in the optimization process, and as discussed below, verification that all other elements of the optimization procedure are accurate to at least eight decimal places.

Critical to the final accuracy of the optimization method of this thesis was the development of routines capable of extremely accurate evaluation of the chi-square cumulative distribution function and its inverse.

Algorithm 299, The Chi-Square Integral of the Association of Computing Machinery Collection was utilized to perform the evaluation of the chi-square cumulative distribution function required by the optimization routine. Algorithm 299 was also employed in a routine which evaluates the inverse of the chi-square cumulative distribution function as required by the optimization process.

For even degrees of freedom, Algorithm 299 is exact. For odd degrees of freedom the accuracy of Algorithm 299 is limited to the accuracy of the normal probability integral which it utilizes to evaluate the lower tail area of the chi-square distribution. Algorithm 304, The Normal Curve Integral of the Association of Computing Machinery Collection was employed

to perform the required lower tail evaluations. The accuracy of Algorithm 304 is machine limited. Thus all evaluations of the chi-square cumulative distribution function and its inverse used in the optimization process are accurate to the limits of the machine used to perform them.

The routines written to evaluate the chi-square cumulative distribution function and its inverse were scrutinized carefully for accuracy. Values computed by both routines were compared with a wide range of sample values taken from Pearsons Table of the Incomplete Gamma Function [4]. In every case both procedures were in precise agreement with the eight decimal tabled values.

The interval of uncertainty algorithm which was employed to solve the nonlinear optimization problems is of arbitrary accuracy, to the limits of the machine. By defining a required final interval of uncertainty it is possible to control the precision with which the true value of the optimal solution will be determined.

Investigation of the characteristics of the objective functions of each of the nonlinear optimization problems revealed that the severity of the nonlinearity was greatest when the level of confidence was high (.99 and above) and the degrees of freedom were simultaneously low. As expected, it was found that in such extreme cases the optimal value of the objective function (and the optimal solution) was quite sensitive to the size of the final interval of uncertainty;

or in essence the precision with which the optimal solution was determined. Experimentation revealed that in even the most severe cases reduction of the final interval of uncertainty below 1×10^{-8} had no effect on the first seven decimal places of the optimal solution. It was decided therefore that a required final interval of 1×10^{-8} would be used in all cases. Thus the "true" optimal value of the decision variable is known only to the extent that it must lie within 5×10^{-9} of the solution found by the optimization routine. This decision is consistent with the fact that the accuracy of the other elements of the optimization procedure could not be verified beyond eight decimal places.

All routines used in the optimization process were programmed in the A.P.L. language and executed on the Naval Postgraduate School I.B.M. 360/67 computer. Under this system, all computations are performed in double-precision (sixteen-digit) arithmetic.

Routine comparison of the results of this thesis with those of Tate and Klett revealed numerous discrepancies between reported values. Without exception, significant discrepancies were confined to levels of confidence of .99 and above. In the vast majority of cases conflict occurred in the third or fourth decimal place. It is worth noting that more than two thirds of the significant discrepancies were found in reported optimal values of C2, for odd degrees of freedom, while fewer than one fourth of the discrepancies

involved optimal values of C1, which was the decision variable in the optimization process used to compute the results of this thesis.

Considerable time and energy were expended in an effort to either reconcile or explain the observed discrepancies. While no firm conclusions were reached, it is felt that the nature of the pattern of discrepancies suggests strongly that the disagreements may be explained in terms of the improved accuracy of Algorithm 299 in evaluating the chi-square integral, and the increased accuracy of present day computer hardware.

By the method utilized in this thesis, once the optimal value of C1 is found by the optimization routine, determination of the optimal value of C2 is simply a matter of evaluating the cumulative distribution function at C1, adding the appropriate amount of probability to the result, and then inverting this result. In view of the general agreement of reported optimal values of C1, and the demonstrated accuracy of Algorithm 299 it is felt that the discrepant results may be attributed to the accuracy advantage which Algorithm 299 enjoys over the numerical integration techniques used for odd degrees of freedom in the earlier work.

It should be noted that where significant discrepancies were found, the tabled values of C1 and C2 were re-checked in order to ensure feasibility with regard to the integral constraint. In every case tested in this manner the feasibility (probability area) constraint was met with at least eight decimal accuracy.

IV. CONCLUSIONS

A. MINIMUM-LENGTH CONFIDENCE INTERVALS

The minimum-length confidence intervals for the variance and standard deviation were said to be optimal because, for a given level of confidence, they are shorter than any other confidence interval based on the sum of the squared deviations of the observations from the mean.

The percent reduction in interval length which is realized when the minimum-length interval for the variance is used in lieu of the equivalent equal-tail interval is depicted graphically in figure 2.

Similarly, figure 3 illustrates the percent reduction in interval length which is realized when the shortest confidence interval for the standard deviation is used in lieu of the equivalent equal-tail interval.

Unfortunately, the reduction in interval length afforded by the use of the minimum-length confidence intervals is not realized without sacrifice. Investigation of the characteristics of the minimum-length confidence intervals/regions of acceptance reveals that they are biased significantly in favor of higher false values of the parameter. Thus, when compared with the unbiased interval, use of the minimum length interval involves a difficult to quantify tradeoff between interval length and biasedness. Figure 1 illustrates vividly the nature of this tradeoff in terms of the increased bias of the minimum-length region of acceptance.

Several additional points regarding the minimum-length interval are worthy of note.

[1] Percent reduction in length of the interval is a function of the degrees of freedom, with savings increasing rapidly with decreasing degrees of freedom.

[2] Percent reduction in interval length is not effected significantly by level of confidence, being essentially constant for a given number of degrees of freedom as the confidence level varies widely.

[3] For very small degrees of freedom, a close approximation to the minimum-length interval will result if all the tail area is placed in the lower tail.

[4] The equal-tail interval exhibits bias toward lower false values of the parameter, while the minimum-length interval exhibits considerably greater bias toward larger false values of the parameter.

[5] "Thumb rules" for the optimal distribution of tail area between the upper and lower tails when constructing the minimum-length intervals may be devised by reference to figures 4 and 5.

B. SHORTEST UNBIASED CONFIDENCE INTERVAL

The shortest unbiased confidence interval was termed optimal, because of all possible confidence intervals of a given level of confidence it is the least likely to cover false values of the parameter. As noted earlier, neither the equal-tail, nor the minimum-length confidence intervals possess this desirable attribute.

In addition, the shortest unbiased confidence interval, although it is of greater length than the minimum-length confidence interval, does offer a significant reduction in confidence interval length when compared with the equivalent equal-tail interval. Figures 6 and 7 illustrate the relationship among the lengths of the equal-tail, minimum-length, and shortest unbiased confidence intervals. Figure 8 illustrates the optimal distribution of complementary tail area between the upper and lower tails for the shortest unbiased confidence intervals, and may be helpful in devising "thumb rules" for the construction of such intervals.

It is worthwhile to note that the shortest unbiased confidence interval possesses an interesting quality of invariance. Thus, the endpoints of the shortest unbiased confidence interval for the standard deviation may be computed by simply taking the positive square root of the endpoints of the shortest unbiased confidence interval for the variance.

C. GENERAL

Two classes of improved/optimal confidence intervals have been proposed, each of which offers significant advantages when compared with the equal-tail confidence intervals. Unfortunately there exists no well-defined measure of effectiveness by which one might establish a ranking between the two improved alternatives. In choosing between the minimum-length interval and the shortest unbiased interval however, the user must recognize the tradeoff in performance inherent in

either choice. To achieve minimum length it is necessary to accept greater bias. Conversely, to eliminate bias it is necessary to accept somewhat greater interval length.

It should be noted also that as the number of degrees of freedom grows arbitrarily large the minimum length, unbiased, and equal-tail confidence intervals converge. Thus, the significance of choosing among the alternatives diminishes as sample size, and hence degrees-of-freedom, increases beyond approximately thirty.

APPENDIX A

MATHEMATICAL DERIVATIONS

It has been shown earlier that in each of the cases of interest it is possible to formulate the problem of solving for the pair (C1,C2) which results in the desired optimal confidence interval as a constrained, nonlinear optimization.

Figures 9, 10, and 11 illustrate typical characteristics of the objective functions $(\frac{1}{C1} - \frac{1}{C2})$, $(\frac{1}{\sqrt{C1}} - \frac{1}{\sqrt{C2}})$, and $(\frac{C2}{C1})$ which are to be minimized in constructing the desired confidence intervals. It can be seen that all three functions are continuous and unimodal over the entire range of feasible values of the decision variable C1.

As a consequence of the desirable characteristics of the objective functions of interest it is possible to determine with arbitrary accuracy that point at which they attain their minimum value through the use of an elementary method of nonlinear optimization, the bisection search, interval of uncertainty algorithm. The bisection search algorithm utilizes objective function derivative information to systematically reduce an initial interval, within which the minimum is known to exist, to an arbitrarily small final interval which contains the minimum. In essence, the algorithm seeks to isolate with arbitrary precision that point at which the objective function derivative vanishes, thus indicating optimality.

In order to effect solution through the use of the bisection search algorithm it is first necessary to derive analytical expressions for the derivatives of the various objective functions with respect to the decision variable C1.

Consider the problem of constructing the minimum-length confidence interval for the variance. It has been shown previously that this problem can be re-formulated as the nonlinear optimization problem:

$$\text{Minimize:} \quad L = \left(\frac{1}{C1} - \frac{1}{C2}\right)$$

$$\text{Subject to:} \quad \int_{0}^{C1} g_K(x)\,dx + \int_{C2}^{\infty} g_K(x)\,dx = 1 - \gamma$$

Differentiation of the objective function with respect to the decision variable C1 yields the expression

$$\frac{\partial L}{\partial C1} = -\left(\frac{1}{C1^2} - \frac{1}{C2^2}\frac{\partial C2}{\partial C1}\right)$$

Application of Leibniz' Rule and the Implicit Function Theorem to the constraint equation yields the following expression for $\frac{\partial C2}{\partial C1}$.

$$\frac{\partial C2}{\partial C1} = \left(\frac{C1}{C2}\right)^{\frac{N-3}{2}} e^{-\frac{1}{2}(C1-C2)}$$

where N is the size of the sample. Substitution and algebraic simplification are all that remain in deriving the desired expression

$$\frac{\partial L}{\partial C1} = \frac{(C1)^{\frac{N+1}{2}} e^{\frac{(C1-C2)}{2}} - (C2)^{\frac{N+1}{2}}}{(C1)^2 (C2)^{\frac{N+1}{2}}}$$

With this expression, it is a simple matter to solve for the values of (C1,C2) which result in the shortest confidence interval for the variance.

The procedure is identical for the shortest confidence interval for the standard deviation, and results in the following expression.

$$\frac{\partial L}{\partial C1} = \frac{(C2)^{-3/2}(C1)^{\frac{N-3}{2}} e^{-\frac{C1}{2}} - (C1)^{-3/2}(C2)^{\frac{N-3}{2}} e^{-\frac{C2}{2}}}{2 \cdot C2^{\frac{N-3}{2}} e^{-\frac{C2}{2}}}$$

In the case of the shortest unbiased confidence interval it is necessary to solve the nonlinear program:

$$\text{Minimize:} \quad L = \frac{C2}{C1}$$

$$\text{Subject to:} \quad \int_0^{C1} g_K(x)dx + \int_{C2}^{\infty} g_K(x)dx = 1 - \gamma$$

As before, derivation of an expression for the derivative of the objective function with respect to the decision variable C1 is simply a matter of differentiation, application of Leibniz' Rule and the Implicit Function Theorem, simple substitution and algebraic simplification. The resultant expression is

$$\frac{\partial L}{\partial C1} = \frac{(C1)^{\frac{N-1}{2}} e^{-\frac{C1}{2}} - (C2)^{\frac{N-1}{2}} e^{-\frac{C2}{2}}}{(C1)^2 (C2)^{\frac{N-3}{2}} e^{-\frac{C2}{2}}}$$

That this method of determining the shortest unbiased confidence interval is equivalent to the classical method of Neyman [2] is easily shown.

In solving the nonlinear optimization problem stated above the bisection search algorithm searches for that point which results in a derivative value of zero, indicating optimality. In order that the derivative vanish, it is necessary that the expression in the numerator approach zero. Hence solution of the nonlinear optimization problem involves finding the pair (C1,C2) which simultaneously, causes the expression

$$(C1)^{\frac{N-1}{2}} e^{-\frac{C1}{2}} - (C2)^{\frac{N-1}{2}} e^{-\frac{C2}{2}}$$

to vanish, and satisfies the constraint equation

$$\int_{C1}^{C2} g_K(x)\,dx = \gamma.$$

Now, consider the classical conditions imposed by Neyman [2] and discussed earlier in this paper. It was shown that in order to construct the shortest unbiased confidence interval it is necessary to find the pair (C1,C2) such that both of the following conditions

$$\int_{C1}^{C2} g_K(x)\,dx = \gamma$$

$$(C1)g_K(C1) - C2g_K(C2) = 0$$

are satisfied. Substitution of the chi-square probability density function into the second of the above classical relations is all that remains to demonstrate that the solution of the nonlinear optimization

$$\text{Minimize:} \quad L = \frac{C2}{C1}$$

$$\text{Subject to:} \quad \int_{C1}^{C2} g_K(x)\,dx = \gamma$$

also satisfies the two conditions imposed by Neyman. Hence the alternative formulations of the problem are equivalent, and result in the same solution.

APPENDIX B

The following tables comprise the principle objective and result of this thesis.

Nine levels of confidence are addressed [.5,.6,.7,.8,.9, .95,.99,.995,.999], for degrees of freedom of one through thirty. For each level of confidence three tables appear: one for the shortest confidence interval for the variance, one for the shortest confidence interval for the standard deviation, and a third for the shortest unbiased confidence interval.

Each table contains the values of C1 and C2 which are to be used in constructing the desired optimal confidence interval. Additionally, the column labeled ($\alpha 1/\alpha 2$) indicates the optimal distribution of complementary tail area between the lower ($\alpha 1$) and upper ($\alpha 2$) tails. The column labeled "REDUCTION" indicates the percent reduction in confidence interval length which is realized when the improved interval is used in lieu of the equivalent equal-tail interval.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.259054	2.543191	3.513828	61.874006
2	0.871103	3.835902	2.403498	35.581319
3	1.576163	5.097337	2.033856	24.606626
4	2.327046	6.329589	1.843343	18.749513
5	3.106942	7.539579	1.724757	15.12961
6	3.907448	8.732345	1.64271	12.676069
7	4.723605	9.911393	1.581968	10.905027
8	5.552193	11.079219	1.534843	9.567152
9	6.390965	12.237661	1.496999	8.521121
10	7.238287	13.388115	1.4658	7.680961
11	8.092921	14.531664	1.439539	6.991425
12	8.953901	15.669171	1.417062	6.415374
13	9.820461	16.801333	1.397554	5.926946
14	10.691975	17.928725	1.380427	5.507577
15	11.567927	19.051825	1.36524	5.143596
16	12.447887	20.171037	1.351659	4.824716
17	13.331487	21.286706	1.339425	4.543047
18	14.218415	22.399129	1.328332	4.292439
19	15.1084	23.508564	1.318215	4.068023
20	16.001207	24.615236	1.308942	3.865898
21	16.896627	25.719343	1.300403	3.682903
22	17.79448	26.821062	1.292509	3.516444
23	18.694601	27.920549	1.285182	3.364378
24	19.596847	29.017945	1.278359	3.224915
25	20.501086	30.113375	1.271985	3.096552
26	21.407202	31.206955	1.266014	2.978014
27	22.315088	32.298787	1.260406	2.868215
28	23.224648	33.388966	1.255127	2.766224
29	24.135794	34.477577	1.250144	2.671236
30	25.048444	35.5647	1.245433	2.582553

CONFIDENCE LEVEL:.5
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.169918	3.061151	3.98848	62.871774
2	0.684425	4.410965	2.629851	36.044252
3	1.315355	5.730384	2.187496	24.847992
4	2.004005	7.016436	1.962476	18.894999
5	2.729111	8.276078	1.823611	15.226317
6	3.480057	9.514951	1.72812	12.744839
7	4.250599	10.737141	1.657754	10.956374
8	5.03669	11.945602	1.603361	9.606919
9	5.835512	13.14255	1.559812	8.552832
10	6.645014	14.329663	1.523998	7.706818
11	7.463649	15.508263	1.493918	7.012912
12	8.290208	16.679401	1.468217	6.433511
13	9.123732	17.843931	1.445949	5.942459
14	9.963439	19.002556	1.426425	5.520995
15	10.808685	20.155865	1.409135	5.155317
16	11.658931	21.304356	1.393691	4.835042
17	12.51372	22.448452	1.379792	4.552213
18	13.372662	23.588521	1.367202	4.300629
19	14.235417	24.724881	1.35573	4.075386
20	15.101692	25.85781	1.345222	3.872553
21	15.971228	26.987553	1.335555	3.688947
22	16.843797	28.114329	1.326621	3.521958
23	17.719197	29.238332	1.318336	3.369428
24	18.597247	30.359734	1.310625	3.229558
25	19.477784	31.478694	1.303426	3.100834
26	20.360662	32.595351	1.296686	2.981977
27	21.245748	33.709835	1.290358	2.871893
28	22.132922	34.822262	1.284403	2.769646
29	23.022073	35.932739	1.278787	2.674428
30	23.913101	37.041364	1.273478	2.585538

CONFIDENCE LEVEL:.6
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.098691	3.731169	4.617361	64.098897
2	0.50616	5.14339	2.926397	36.678546
3	1.052329	6.527284	2.386282	25.187096
4	1.669224	7.874108	2.11526	19.10144
5	2.330956	9.190316	1.949568	15.364205
6	3.024486	10.48199	1.836413	12.843179
7	3.742131	11.75375	1.753471	11.029936
8	4.478906	13.009018	1.689625	9.663982
9	5.231343	14.250388	1.638682	8.598357
10	5.996921	15.479854	1.59691	7.743973
11	6.77373	16.698989	1.561911	7.043805
12	7.560288	17.909053	1.532072	6.459599
13	8.355411	19.11107	1.506266	5.964779
14	9.15814	20.305891	1.483678	5.540308
15	9.967683	21.494226	1.463705	5.172191
16	10.783375	22.676676	1.445888	4.849911
17	11.604655	23.853757	1.429872	4.565414
18	12.431042	25.025914	1.41538	4.312428
19	13.262121	26.193533	1.402189	4.085994
20	14.097528	27.356953	1.390119	3.882142
21	14.936948	28.516473	1.379022	3.697657
22	15.780099	29.672358	1.368777	3.529904
23	16.626731	30.824846	1.359282	3.376706
24	17.476624	31.974147	1.350451	3.236249
25	18.329576	33.120453	1.342212	3.107008
26	19.185407	34.263937	1.334502	2.987689
27	20.043955	35.404755	1.327269	2.877194
28	20.905072	36.54305	1.320466	2.774579
29	21.768621	37.678953	1.314052	2.67903
30	22.634479	38.812583	1.307992	2.589841

CONFIDENCE LEVEL:.7
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.045737	4.672214	5.524374	65.612316
2	0.3346	6.160583	3.352949	37.587606
3	0.778863	7.621573	2.668664	25.692009
4	1.307761	9.042031	2.330145	19.413482
5	1.891152	10.427411	2.12539	15.574242
6	2.513412	11.784043	1.986696	12.993653
7	3.16523	13.11703	1.885676	11.142808
8	3.840529	14.430276	1.808312	9.751705
9	4.535073	15.726797	1.746849	8.668461
10	5.245777	17.008926	1.696628	7.801245
11	5.970298	18.278532	1.65468	7.091466
12	6.706818	19.537114	1.619012	6.499874
13	7.453884	20.785899	1.588238	5.999258
14	8.210316	22.025907	1.561357	5.570156
15	8.975141	23.257996	1.537632	5.198279
16	9.747544	24.482892	1.516503	4.872907
17	10.526832	25.701219	1.497541	4.585837
18	11.312415	26.913516	1.480406	4.330685
19	12.10378	28.120253	1.464829	4.102413
20	12.900483	29.321841	1.450593	3.896987
21	13.702133	30.518646	1.437519	3.711143
22	14.508383	31.71099	1.425461	3.542209
23	15.318928	32.899162	1.414296	3.38798
24	16.133493	34.08342	1.403922	3.246615
25	16.951832	35.263999	1.394251	3.116571
26	17.773723	36.441109	1.385208	2.99654
27	18.598965	37.614941	1.37673	2.885409
28	19.427375	38.785671	1.368762	2.782224
29	20.258787	39.953459	1.361255	2.686163
30	21.093049	41.11845	1.354167	2.596511

CONFIDENCE LEVEL:.8
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.012116	6.259471	7.095128	67.574482
2	0.16763	7.864292	4.101634	39.062701
3	0.476395	9.433824	3.158716	26.567415
4	0.882654	10.958349	2.698756	19.969457
5	1.354693	12.442346	2.424137	15.953795
6	1.87459	13.892227	2.240074	13.267729
7	2.431261	15.313562	2.107173	11.349521
8	3.017327	16.710795	2.006121	9.912928
9	3.627588	18.087432	1.92632	8.797599
10	4.258219	19.446252	1.861451	7.906991
11	4.906311	20.789488	1.807508	7.179605
12	5.569586	22.118958	1.76182	6.574447
13	6.246227	23.436159	1.722534	6.063163
14	6.934751	24.742344	1.688325	5.625523
15	7.633933	26.038568	1.658214	5.246708
16	8.342743	27.32573	1.631466	4.915622
17	9.06031	28.604604	1.607514	4.623791
18	9.785886	29.87586	1.585916	4.364631
19	10.518823	31.140086	1.566319	4.132952
20	11.258557	32.397795	1.54844	3.924607
21	12.004594	33.649444	1.532047	3.736243
22	12.756494	34.895437	1.516951	3.565119
23	13.51387	36.136137	1.502994	3.408974
24	14.276372	37.37187	1.490041	3.265924
25	15.043688	38.602929	1.477982	3.134389
26	15.815537	39.82958	1.46672	3.013033
27	16.591663	41.052064	1.456173	2.90072
28	17.371832	42.270603	1.446269	2.796476
29	18.155835	43.485398	1.436949	2.699461
30	18.943476	44.696635	1.428157	2.608949

CONFIDENCE LEVEL:.9
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.003159	7.816846	8.659476	68.921715
2	0.084727	9.530336	4.867543	40.35848
3	0.296238	11.191472	3.657911	27.407172
4	0.607001	12.802444	3.070821	20.5236
5	0.989227	14.368614	2.723013	16.339644
6	1.425002	15.896592	2.491646	13.549624
7	1.902588	17.39226	2.325659	11.563619
8	2.41392	18.860434	2.200178	10.080881
9	2.953213	20.304952	2.101558	8.93265
10	3.516159	21.728898	2.021721	8.017884
11	4.099444	23.134795	1.955578	7.272238
12	4.700465	24.524694	1.899739	6.652962
13	5.31713	25.900303	1.851864	6.130544
14	5.947728	27.263051	1.810283	5.683971
15	6.590839	28.61415	1.77377	5.297883
16	7.245268	29.954637	1.741403	4.960798
17	7.910001	31.285406	1.712477	4.66396
18	8.584165	32.607233	1.686439	4.400581
19	9.267005	33.920799	1.662853	4.165313
20	9.957862	35.2267	1.641367	3.95389
21	10.656156	36.525469	1.621696	3.762866
22	11.361373	37.817578	1.603604	3.589428
23	12.073056	39.103448	1.586897	3.431257
24	12.790798	40.383462	1.571412	3.286425
25	13.514231	41.657961	1.557009	3.153313
26	14.243024	42.927259	1.543573	3.030555
27	14.976875	44.191636	1.531001	2.91699
28	15.715512	45.451354	1.519208	2.811623
29	16.458685	46.706647	1.508118	2.713598
30	17.206166	47.957735	1.497667	2.622173

CONFIDENCE LEVEL:.95
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	*C*-1	*C*-2	$\alpha 1/\alpha 2$	*REDUCTION*
1	0.000134	11.3449	12.217821	70.741268
2	0.017469	13.285447	6.671814	42.63375
3	0.101048	15.126949	4.843463	29.100408
4	0.263963	16.90132	3.950088	21.718251
5	0.496234	18.621307	3.423702	17.202404
6	0.785646	20.295553	3.076581	14.194247
7	1.122115	21.930877	2.829867	12.060723
8	1.497847	23.532764	2.644949	10.474499
9	1.90684	25.105648	2.500776	9.251523
10	2.344412	26.65313	2.384907	8.281129
11	2.806854	28.178171	2.289526	7.493082
12	3.291176	29.68322	2.209469	6.840787
13	3.794933	31.170332	2.14119	6.29218
14	4.3161	32.641238	2.08217	5.824499
15	4.852976	34.097421	2.030568	5.421161
16	5.404116	35.540151	1.985009	5.069801
17	5.968285	36.97053	1.94444	4.76102
18	6.544414	38.389523	1.908046	4.487553
19	7.131572	39.797974	1.875181	4.243685
20	7.728943	41.196631	1.84533	4.024873
21	8.335805	42.586157	1.818073	3.827456
22	8.95152	43.967145	1.793069	3.648448
23	9.575514	45.340129	1.770034	3.485397
24	10.207277	46.705587	1.74873	3.336263
25	10.846345	48.063953	1.728958	3.199342
26	11.492301	49.415623	1.710548	3.073196
27	12.144765	50.760954	1.693356	2.956602
28	12.803392	52.100275	1.677257	2.848517
29	13.467864	53.433888	1.662144	2.748044
30	14.137891	54.76207	1.647923	2.654407

CONFIDENCE LEVEL:.99
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000034	12.838765	13.727029	71.225157
2	0.008836	14.864682	7.448791	43.351753
3	0.063893	16.775259	5.360681	29.708533
4	0.185933	18.610253	4.334352	22.178659
5	0.37234	20.386419	3.72895	17.548503
6	0.614451	22.113828	3.330192	14.459262
7	0.903756	23.799896	3.047361	12.268377
8	1.23315	25.450439	2.835869	10.640823
9	1.596916	27.07012	2.671361	9.387366
10	1.990473	28.662726	2.539446	8.393972
11	2.410117	30.231368	2.43108	7.588202
12	2.852827	31.778629	2.340301	6.921994
13	3.316104	33.306678	2.263015	6.362279
14	3.797861	34.817349	2.196319	5.8856
15	4.296334	36.312213	2.138094	5.474876
16	4.81002	37.792621	2.086758	5.117382
17	5.337625	39.259746	2.041106	4.803456
18	5.878027	40.714614	2.0002	4.525626
19	6.430245	42.158127	1.963303	4.278032
20	6.993417	43.591083	1.929824	4.056015
21	7.566781	45.014193	1.899286	3.855819
22	8.149657	46.428092	1.871297	3.674386
23	8.741439	47.833349	1.845534	3.509207
24	9.341581	49.23048	1.821727	3.358197
25	9.949591	50.619949	1.799649	3.219612
26	10.565023	52.00218	1.779107	3.091983
27	11.187472	53.37756	1.759937	2.974064
28	11.816568	54.74644	1.741998	2.864788
29	12.451973	56.109146	1.725167	2.763243
30	13.093375	57.465975	1.70934	2.668634

CONFIDENCE LEVEL:.995
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000001	16.241693	16.936121	71.960571
2	0.001805	18.467697	9.23786	44.600406
3	0.022097	20.523858	6.565454	30.868614
4	0.083097	22.485574	5.234476	23.114017
5	0.19336	24.377715	4.444585	18.280069
6	0.352026	26.214113	3.923826	15.034006
7	0.554914	28.003917	3.555116	12.726641
8	0.797223	29.75386	3.280239	11.012427
9	1.074459	31.469185	3.067183	9.693625
10	1.382682	33.154089	2.896961	8.650126
11	1.71852	34.812005	2.757631	7.80528
12	2.0791	36.445792	2.641317	7.108107
13	2.461975	38.057838	2.542616	6.523489
14	2.865051	39.650182	2.457698	6.026517
15	3.286528	41.224568	2.38378	5.599051
16	3.724848	42.782497	2.318784	5.227603
17	4.178652	44.325283	2.261129	4.901922
18	4.646751	45.854073	2.20959 1	4.614107
19	5.128098	47.36988	2.163207	4.357964
20	5.621764	48.873603	2.121207	4.128569
21	6.126922	50.366041	2.082971	3.921955
22	6.642831	51.847913	2.047993	3.734931
23	7.168829	53.319857	2.015852	3.564828
24	7.704315	54.782466	1.9862	3.40947
25	8.248746	56.236258	1.958745	3.267028
26	8.801629	57.681721	1.933237	3.135958
27	9.362512	59.119289	1.909467	3.014956
28	9.930985	60.549368	1.887253	2.902913
29	10.506668	61.972324	1.866438	2.798867
30	11.089214	63.388499	1.846888	2.702

CONFIDENCE LEVEL:.999
SHORTEST UNBIASED CONFIDENCE INTERVAL.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.450903	9.637354	261.247324	76.75256
2	1.340623	8.922299	42.293485	53.981189
3	2.239808	9.424332	19.703833	41.450017
4	3.128972	10.246168	12.706509	33.623205
5	4.012297	11.195055	9.493869	28.279021
6	4.89347	12.205065	7.688499	24.399556
7	5.774659	13.247893	6.542162	21.45566
8	6.657051	14.309571	5.752656	19.145472
9	7.541276	15.382521	5.176609	17.284319
10	8.427647	16.462339	4.737873	15.752905
11	9.316297	17.546337	4.392472	14.470746
12	10.207256	18.632803	4.113324	13.381576
13	11.100498	19.720621	3.882869	12.444876
14	11.995962	20.809042	3.689241	11.630727
15	12.893571	21.89756	3.524139	10.916556
16	13.793237	22.985824	3.381583	10.285014
17	14.694873	24.073594	3.257162	9.722545
18	15.59839	25.160705	3.147547	9.218405
19	16.503702	26.247043	3.05018	8.763969
20	17.410726	27.332533	2.963065	8.352232
21	18.319383	28.417126	2.884617	7.977445
22	19.229601	29.500794	2.813567	7.634849
23	20.141307	30.58352	2.748882	7.320466
24	21.054437	31.665301	2.689717	7.030951
25	21.968928	32.74614	2.635368	6.763463
26	22.884723	33.826044	2.58525	6.515583
27	23.801767	34.905027	2.538869	6.285229
28	24.720009	35.983103	2.495806	6.070608
29	25.639401	37.06029	2.455703	5.870159
30	26.559897	38.136607	2.418253	5.682525

CONFIDENCE LEVEL:.5
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.273936	11.480449	567.73006	76.198371
2	1.002729	10.333323	69.131417	53.34279
3	1.80678	10.700503	28.716199	40.978851
4	2.628462	11.466809	17.3574	33.282252
5	3.457847	12.393568	12.43393	28.025501
6	4.292736	13.398461	9.773798	24.205123
7	5.1325	14.445793	8.132975	21.302378
8	5.976845	15.517784	7.027982	19.021777
9	6.825532	16.604707	6.236115	17.18252
10	7.678331	17.70087	5.641858	15.667727
11	8.53501	18.802769	5.179796	14.398462
12	9.395342	19.908164	4.8103	13.319485
13	10.259111	21.015578	4.508042	12.390978
14	11.12611	22.124016	4.256115	11.583508
15	11.996147	23.232798	4.042822	10.874854
16	12.869045	24.34145	3.859819	10.247918
17	13.74464	25.449643	3.701001	9.689336
18	14.622779	26.557146	3.561799	9.188504
19	15.503322	27.6638	3.438728	8.736906
20	16.386141	28.769493	3.329084	8.327622
21	17.271117	29.874153	3.230736	7.954971
22	18.15814	30.977733	3.141984	7.614244
23	19.04711	32.080204	3.061454	7.301507
24	19.937932	33.181553	2.988023	7.013448
25	20.83052	34.281778	2.920767	6.747256
26	21.724793	35.380884	2.858914	6.500532
27	22.620676	36.47888	2.801818	6.271216
28	23.518101	37.575783	2.748933	6.057528
29	24.417001	38.671609	2.699795	5.857923
30	25.317316	39.766379	2.654003	5.671054

CONFIDENCE LEVEL:.6
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.148253	13.734474	1423.812332	75.71645
2	0.706468	12.053946	123.337552	52.629637
3	1.395707	12.260696	44.861417	40.406318
4	2.132154	12.861051	25.160471	32.849576
5	2.892735	13.860742	17.161318	27.695155
6	3.668837	14.858317	13.026701	23.947238
7	4.456445	15.90957	10.558243	21.096485
8	5.253325	16.992338	8.937405	18.854047
9	6.058057	18.09441	7.799232	17.043469
10	6.869639	19.208593	6.959282	15.550699
11	7.687313	20.330429	6.31533	14.298677
12	8.510478	21.45705	5.806561	13.233435
13	9.33864	22.586549	5.394709	12.316034
14	10.171381	23.717636	5.054576	11.517668
15	11.008344	24.849417	4.768937	10.816564
16	11.84922	25.981268	4.525636	10.195958
17	12.693736	27.112749	4.315865	9.642732
18	13.541651	28.243548	4.133089	9.146473
19	14.39275	29.373446	3.972362	8.698809
20	15.24684	30.502286	3.829874	8.292933
21	16.103747	31.629965	3.702646	7.923254
22	16.963314	32.756409	3.588311	7.585134
23	17.825398	33.881573	3.48497	7.274696
24	18.689868	35.005432	3.39108	6.988675
25	19.556603	36.127974	3.305373	6.724297
26	20.425495	37.249198	3.226799	6.47916
27	21.296441	38.369111	3.154484	6.251336
28	22.169349	39.487726	3.087687	6.038961
29	23.04413	40.60506	3.025784	5.840543
30	23.920706	41.721134	2.968241	5.65475

CONFIDENCE LEVEL:.7
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.064157	16.760619	4715.500084	75.337547
2	0.444366	14.341396	259.150411	51.845286
3	0.994905	14.33341	79.49181	39.703898
4	1.622109	14.946682	40.553356	32.287977
5	2.292228	15.810102	25.998994	27.251178
6	2.990326	16.796748	18.880759	23.592328
7	3.708633	17.851402	14.799358	20.808218
8	4.442585	18.946315	12.201837	18.616143
9	5.18925	20.066142	10.422974	16.844229
10	5.9466	21.201795	9.137072	15.381642
11	6.713154	22.34761	8.168256	14.153564
12	7.487786	23.499919	7.414161	13.107598
13	8.269607	24.656272	6.811579	12.205923
14	9.057895	25.814996	6.319567	11.420541
15	9.852056	26.974924	5.910534	10.730274
16	10.651588	28.135235	5.565264	10.118803
17	11.456063	29.295342	5.26999	9.573345
18	12.26511	30.454828	5.014606	9.083744
19	13.078406	31.61339	4.791536	8.64183
20	13.895667	32.770815	4.594997	8.240952
21	14.716639	33.926951	4.420499	7.875643
22	15.541097	35.081693	4.264505	7.541367
23	16.368838	36.23497	4.124195	7.234328
24	17.199678	37.386737	3.997292	6.951326
25	18.033453	38.536969	3.881939	6.689642
26	18.870009	39.685657	3.776605	6.446953
27	19.70921	40.832799	3.680019	6.221264
28	20.550926	41.978406	3.591116	6.010848
29	21.395043	43.122492	3.508998	5.814204
30	22.241452	44.265078	3.4329	5.630023

CONFIDENCE LEVEL:.8
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.01579	21.69442	31276.70364	75.08952
2	0.21045	18.00772	812.44291	50.98362
3	0.58208	17.63813	190.46997	38.79262
4	1.05608	18.10623	83.99974	31.49544
5	1.59379	18.90813	49.01619	26.59119
6	2.17505	19.87391	33.29153	23.04557
7	2.78826	20.93027	24.80818	20.35238
8	3.4262	22.04051	19.65528	18.23235
9	4.08403	23.18436	16.25542	16.51769
10	4.75836	24.34981	13.87186	15.10101
11	5.4467	25.52933	12.1217	13.91012
12	6.14717	26.71803	10.78913	12.89461
13	6.85827	27.91264	9.74448	12.01814
14	7.57882	29.1109	8.90573	11.25382
15	8.30787	30.31122	8.21876	10.58132
16	9.0446	31.51248	7.64657	9.98495
17	9.78835	32.71386	7.16308	9.45244
18	10.53853	33.91479	6.74945	8.97401
19	11.29465	35.11483	6.39174	8.5418
20	12.05627	36.31368	6.07943	8.14941
21	12.82302	37.5111	5.80447	7.79156
22	13.59456	38.70695	5.56057	7.46387
23	14.37058	39.9011	5.34278	7.16268
24	15.15082	41.09349	5.14711	6.88489
25	15.93505	42.28406	4.97036	6.62787
26	16.72303	43.47278	4.80992	6.38938
27	17.51458	44.65965	4.66361	6.16747
28	18.3095	45.84466	4.52963	5.96048
29	19.10765	47.02782	4.40649	5.76694
30	19.90885	48.20915	4.2929	5.58559

CONFIDENCE LEVEL:.9
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.003932	26.468167	186564.801965	75.023124
2	0.102541	21.481323	2308.823614	50.515252
3	0.351247	20.743831	419.197816	38.176192
4	0.70825	21.063149	161.51224	30.899342
5	1.139245	21.800077	86.496348	26.062035
6	1.623295	22.740977	55.303828	22.587768
7	2.147323	23.794365	39.383273	19.958418
8	2.702723	24.914707	30.111175	17.892513
9	3.283552	26.076931	24.193946	16.222953
10	3.885528	27.266194	20.156954	14.843717
11	4.50546	28.473225	17.259565	13.684012
12	5.140899	29.691989	15.095987	12.694613
13	5.78992	30.918429	13.428213	11.840149
14	6.450975	32.149749	12.108798	11.094508
15	7.1228	33.383978	11.042199	10.437965
16	7.804345	34.619711	10.164142	9.85533
17	8.494727	35.855932	9.429987	9.334703
18	9.193192	37.091899	8.80788	8.866626
19	9.899095	38.327069	8.274544	8.443481
20	10.611873	39.561044	7.812618	8.059064
21	11.331035	40.793531	7.40891	7.708267
22	12.056147	42.024319	7.05324	7.386847
23	12.786824	43.253254	6.737631	7.091248
24	13.522723	44.480232	6.455757	6.818469
25	14.263535	45.70518	6.202537	6.565957
26	15.008982	46.928052	5.973849	6.331529
27	15.758814	48.148824	5.766316	6.113303
28	16.512799	49.367487	5.577148	5.909652
29	17.27073	50.584043	5.404018	5.71916
30	18.032415	51.798506	5.244973	5.540588

CONFIDENCE LEVEL:.95
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000157	28.113019	87409.184164	75.000425
2	0.0201	29.127441	21131.089998	50.110477
3	0.114796	27.510787	2172.683501	37.433448
4	0.296859	27.460335	622.341108	30.053038
5	0.553433	28.026878	277.078359	25.23334
6	0.870012	28.892716	155.868164	21.821912
7	1.235019	29.922836	100.931899	19.267025
8	1.639725	31.050629	71.665013	17.273847
9	2.077549	32.23963	54.249014	15.670537
10	2.54346	33.468366	43.022302	14.349895
11	3.033537	34.723459	35.336092	13.241364
12	3.544673	35.996227	29.821752	12.296453
13	4.074364	37.280834	25.715295	11.480652
14	4.620561	38.573247	22.563069	10.768673
15	5.181562	39.870629	20.081807	10.141525
16	5.755943	41.170938	18.087073	9.584654
17	6.342492	42.472694	16.454467	9.086694
18	6.940173	43.774809	15.097487	8.638644
19	7.548089	45.076478	13.954418	8.23326
20	8.165459	46.377104	12.980206	7.864655
21	8.791599	47.67624	12.141295	7.527988
22	9.425904	48.973557	11.412247	7.21924
23	10.067836	50.268812	10.773474	6.935046
24	10.716917	51.561829	10.209674	6.672563
25	11.372715	52.85248	9.708743	6.429375
26	12.034842	54.14068	9.260993	6.203416
27	12.702946	55.426371	8.858581	5.992904
28	13.376709	56.70952	8.495111	5.796299
29	14.055839	57.990112	8.165309	5.612257
30	14.740069	59.268148	7.864795	5.439605

CONFIDENCE LEVEL:.99
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000039	26.771726	21835.213659	74.997965
2	0.010025	32.284375	51213.040567	50.05632
3	0.07171	30.303166	4195.325659	37.280711
4	0.206888	30.084321	1061.779423	29.840904
5	0.411343	30.569665	437.985592	25.000905
6	0.674679	31.396368	233.823252	21.590918
7	0.987092	32.410317	145.695309	19.047543
8	1.340586	33.535591	100.419401	17.069795
9	1.72885	34.730693	74.228541	15.482814
10	2.146907	35.97119	57.726053	14.177998
11	2.590796	37.241895	46.639148	13.084185
12	3.057325	38.532951	38.810841	12.152684
13	3.543889	39.837738	33.060532	11.348969
14	4.048337	41.151672	28.698709	10.647828
15	4.568868	42.471536	25.301154	10.030381
16	5.103963	43.79501	22.595166	9.482187
17	5.652326	45.120409	20.398869	8.992005
18	6.212842	46.446493	18.587118	8.550923
19	6.784543	47.772339	17.0714	8.151807
20	7.366585	49.097264	15.787655	7.788854
21	7.958224	50.420757	14.688557	7.45729
22	8.558802	51.742433	13.738465	7.153165
23	9.167735	53.062009	12.910105	6.873167
24	9.784498	54.379279	12.182314	6.614504
25	10.408623	55.694086	11.53843	6.374802
26	11.039683	57.006331	10.965198	6.152029
27	11.677295	58.315939	10.451938	5.944439
28	12.321109	59.622864	9.989979	5.750517
29	12.970805	60.927085	9.572204	5.568946
30	13.626091	62.228592	9.192721	5.398572

CONFIDENCE LEVEL:.995
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000002	23.192148	681.148928	74.926545
2	0.002001	40.2	511175.895187	50.011607
3	0.024297	36.604562	17953.668243	37.102528
4	0.090791	35.985106	3430.115415	29.546413
5	0.21014	36.266445	1189.362377	24.643465
6	0.380831	36.989064	564.091712	21.211094
7	0.597927	37.953872	322.408114	18.669196
8	0.855983	39.062678	207.968141	16.705422
9	1.150001	40.262101	145.822077	15.138277
10	1.475664	41.52066	108.605426	13.855467
11	1.829314	42.818795	84.626598	12.783851
12	2.207872	44.1438	68.28177	11.873731
13	2.608738	45.487213	56.633129	11.090097
14	3.029706	46.843121	48.025695	10.407553
15	3.468892	48.207424	41.473854	9.807175
16	3.924679	49.577059	36.360777	9.274592
17	4.395668	50.949917	32.285991	8.798636
18	4.88064	52.324356	28.979549	8.370523
19	5.378527	53.69917	26.254337	7.983228
20	5.888389	55.073476	23.977479	7.63105
21	6.409394	56.446583	22.052365	7.309317
22	6.940799	57.817997	20.407205	7.014198
23	7.481941	59.187327	18.988161	6.742436
24	8.032223	60.554257	17.753551	6.491333
25	8.591108	61.918627	16.67141	6.258576
26	9.158108	63.280253	15.716198	6.042195
27	9.73278	64.639018	14.867847	5.840502
28	10.314721	65.99487	14.110083	5.652026
29	10.90356	67.34774	13.42965	5.475492
30	11.498958	68.697615	12.815753	5.30979

CONFIDENCE LEVEL:.999
SHORTEST CONFIDENCE INTERVAL FOR THE VARIANCE.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.418187	5.611466	27.021675	50.453544
2	1.201178	6.052863	9.311755	29.510475
3	2.004718	7.014784	6.000052	20.750785
4	2.819995	8.088595	4.65699	15.986855
5	3.646054	9.19828	3.930382	12.998584
6	4.481733	10.320779	3.473184	10.950491
7	5.325857	11.447297	3.157546	9.459569
8	6.17738	12.574044	2.925567	8.325811
9	7.035418	13.699279	2.747234	7.434656
10	7.899226	14.8222	2.605427	6.715786
11	8.768182	15.942461	2.489659	6.123654
12	9.64176	17.059947	2.393139	5.627465
13	10.519516	18.174667	2.311272	5.205652
14	11.401069	19.286692	2.240835	4.842659
15	12.286092	20.396123	2.179494	4.526987
16	13.174301	21.503078	2.125522	4.249949
17	14.065447	22.607675	2.077607	4.004861
18	14.959314	23.710035	2.034738	3.786499
19	15.855708	24.810272	1.996118	3.590717
20	16.754458	25.908496	1.961115	3.414185
21	17.655412	27.004809	1.929218	3.254198
22	18.558433	28.099309	1.900008	3.108532
23	19.463396	29.192085	1.873141	2.975349
24	20.37019	30.283221	1.848332	2.853108
25	21.278714	31.372795	1.825338	2.740516
26	22.188875	32.460881	1.803957	2.636472
27	23.100588	33.547547	1.784014	2.54004
28	24.013776	34.632854	1.76536	2.450413
29	24.928367	35.716864	1.747867	2.366895
30	25.844296	36.79963	1.731422	2.288883

CONFIDENCE LEVEL:.5
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.261248	6.771167	42.176847	50.355262
2	0.923452	7.00016	12.247243	29.468437
3	1.652116	7.922937	7.397802	20.727814
4	2.408375	8.999407	5.545114	15.972376
5	3.183468	10.126684	4.576914	12.988621
6	3.973262	11.273062	3.981633	10.943216
7	4.775123	12.426263	3.5775	9.454022
8	5.587165	13.58089	3.284266	8.321442
9	6.407961	14.734397	3.061128	7.431127
10	7.23639	15.885556	2.885166	6.712875
11	8.071555	17.033789	2.742512	6.121212
12	8.912719	18.178857	2.62428	5.625387
13	9.759274	19.320703	2.52451	5.203862
14	10.610704	20.459366	2.439053	4.841102
15	11.466574	21.594938	2.364927	4.52562
16	12.326507	22.727537	2.299936	4.248738
17	13.190178	23.857293	2.242421	4.003783
18	14.057303	24.984338	2.191111	3.785532
19	14.927631	26.108804	2.145007	3.589845
20	15.800942	27.230816	2.103321	3.413394
21	16.677038	28.350495	2.065417	3.253477
22	17.555744	29.467955	2.030777	3.107873
23	18.4369	30.583301	1.998977	2.974743
24	19.320365	31.696635	1.969662	2.852551
25	20.206008	32.808051	1.942537	2.740001
26	21.093711	33.917634	1.917353	2.635995
27	21.983367	35.025469	1.893897	2.539596
28	22.874878	36.13163	1.871986	2.449999
29	23.768152	37.236191	1.851465	2.366508
30	24.663106	38.339217	1.832197	2.28852 1

CONFIDENCE LEVEL:.6
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.144229	8.233318	71.942542	50.258588
2	0.666417	8.194223	17.049871	29.420391
3	1.302693	9.060778	9.528667	20.699487
4	1.986567	10.134015	6.847296	15.953775
5	2.699826	11.277627	5.501245	12.9755
6	3.434432	12.448894	4.695642	10.933475
7	4.185786	13.631008	4.159321	9.446511
8	4.950866	14.816439	3.775967	8.315476
9	5.727519	16.001552	3.487713	7.426274
10	6.514126	17.184524	3.262614	6.708852
11	7.309428	18.364446	3.081611	6.117823
12	8.112419	19.540888	2.932639	5.622493
13	8.922276	20.713686	2.807685	5.201363
14	9.738316	21.882821	2.70122	4.838922
15	10.559966	23.048357	2.609303	4.523701
16	11.386734	24.210401	2.529047	4.247037
17	12.218199	25.369083	2.458291	4.002263
18	13.053995	26.524544	2.39538	3.784167
19	13.893802	27.676928	2.339028	3.588612
20	14.737336	28.826375	2.28822	3.412276
21	15.584348	29.973022	2.24214	3.252457
22	16.434613	31.116999	2.200131	3.10694
23	17.287931	32.258429	2.16165	2.973886
24	18.144121	33.397429	2.126251	2.85176
25	19.003021	34.534109	2.093559	2.739269
26	19.864481	35.668572	2.063261	2.635316
27	20.728367	36.800914	2.03509	2.538965
28	21.594554	37.931227	2.008817	2.44941
29	22.462929	39.059596	1.984248	2.365958
30	23.333388	40.186099	1.961212	2.288005

CONFIDENCE LEVEL:.7
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.063285	10.235175	144.150961	50.16893
2	0.428032	9.830033	26.264312	29.368045
3	0.947222	10.61208	13.265788	20.66497
4	1.537614	11.672731	9.020885	15.929633
5	2.170998	12.831113	6.995561	12.957804
6	2.834546	14.029518	5.824037	10.920003
7	3.521098	15.244848	5.063215	9.435934
8	4.226107	16.466561	4.529643	8.306963
9	4.946436	17.689444	4.134478	7.41928
10	5.679789	18.910799	3.829704	6.703007
11	6.424416	20.129212	3.587176	6.112867
12	7.178945	21.343955	3.389333	5.618239
13	7.942266	22.554684	3.22466	5.197672
14	8.71347	23.761276	3.085295	4.83569
15	9.4918	24.963734	2.965688	4.520847
16	10.276612	26.162134	2.861811	4.244499
17	11.067358	27.356593	2.770666	3.999992
18	11.863564	28.547251	2.689979	3.782122
19	12.664816	29.734258	2.617989	3.586762
20	13.470751	30.917766	2.553314	3.410593
21	14.281048	32.097925	2.494855	3.250921
22	15.095422	33.274881	2.441722	3.105531
23	15.913619	34.448775	2.393191	2.97259
24	16.735407	35.61974	2.348664	2.850564
25	17.560581	36.787904	2.307646	2.738161
26	18.388952	37.953387	2.269718	2.634288
27	19.22035	39.116301	2.23453	2.538007
28	20.054618	40.276754	2.20178	2.448517
29	20.891613	41.434846	2.171213	2.365122
30	21.731205	42.590671	2.142605	2.287222

CONFIDENCE LEVEL:.8
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.015716	13.531937	425.370464	50.088448
2	0.206481	12.521445	51.35972	29.31602
3	0.565351	13.153266	22.16631	20.62346
4	1.02004	14.17999	13.831925	15.897004
5	1.535236	15.349709	10.150673	12.932123
6	2.092945	16.580552	8.128323	10.899516
7	2.682771	17.839118	6.863277	9.419318
8	3.298085	19.109834	6.001251	8.293265
9	3.934341	20.384804	5.377309	7.407819
10	4.588249	21.659807	4.905057	6.69329
11	5.25733	22.932525	4.535138	6.104532
12	5.939654	24.201678	4.23741	5.611016
13	6.633683	25.466579	3.99247	5.191354
14	7.338161	26.726889	3.78728	4.83012
15	8.052051	27.982479	3.612766	4.515901
16	8.774478	29.233343	3.462421	4.240078
17	9.504701	30.479553	3.331458	3.996017
18	10.24208	31.721225	3.216281	3.77853
19	10.98606	32.958499	3.114134	3.5835
20	11.736158	34.191532	3.02287	3.407618
21	12.491945	35.420481	2.940791	3.248196
22	13.253042	36.645508	2.866539	3.103027
23	14.019112	37.866769	2.79901	2.970281
24	14.789851	39.084418	2.737304	2.848428
25	15.564986	40.298599	2.680672	2.73618
26	16.344269	41.509455	2.628492	2.632444
27	17.127476	42.717116	2.58024	2.536288
28	17.914403	43.921712	2.535472	2.44691
29	18.704861	45.12336	2.493809	2.363617
30	19.498679	46.322175	2.454927	2.285808

CONFIDENCE LEVEL:.9
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.003925	16.717698	1151.751244	50.049187
2	0.101486	15.111333	94.577184	29.295128
3	0.344889	15.589382	35.327641	20.601193
4	0.691777	16.573154	20.376125	15.875614
5	1.109204	17.743444	14.217283	12.913209
6	1.577644	18.99555	10.987088	10.88328
7	2.085048	20.286266	9.033239	9.405476
8	2.623487	21.59518	7.735668	8.281438
9	3.187437	22.911776	6.815523	7.397653
10	3.772872	24.230324	6.130678	6.684488
11	4.376749	25.547591	5.601731	6.096854
12	4.996697	26.861727	5.181091	5.60427
13	5.630827	28.17168	4.83861	5.185388
14	6.2776	29.476875	4.554309	4.824807
15	6.935744	30.777028	4.314453	4.511144
16	7.604192	32.072032	4.109296	4.235796
17	8.282037	33.361895	3.931742	3.992144
18	8.9685	34.646693	3.776503	3.77501
19	9.662907	35.926545	3.63956	3.580287
20	10.364668	37.201595	3.517804	3.404675
21	11.073265	38.472001	3.408795	3.245491
22	11.788238	39.737924	3.31059	3.100532
23	12.509178	40.99953	3.221622	2.967972
24	13.235717	42.256981	3.140616	2.846286
25	13.967527	43.510434	3.06652	2.734188
26	14.704307	44.760042	2.998464	2.630586
27	15.445788	46.005952	2.935716	2.534552
28	16.191722	47.248305	2.87766	2.445283
29	16.941884	48.487233	2.823773	2.36209
30	17.696066	49.722864	2.773606	2.284373

CONFIDENCE LEVEL:.95
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000157	23.871207	9707.620613	50.012498
2	0.020041	20.86405	338.290775	29.29051
3	0.113995	20.973416	92.788465	20.592654
4	0.293721	21.837496	45.314974	15.858004
5	0.546055	22.985427	28.434424	12.891669
6	0.856671	24.261944	20.396311	10.861216
7	1.214335	25.602184	15.861003	9.384462
8	1.610672	26.975006	13.004698	8.262065
9	2.039406	28.364042	11.062755	7.380054
10	2.495749	29.760143	9.666346	6.668595
11	2.975969	31.157976	8.61845	6.082525
12	3.477102	32.55436	7.805331	5.59134
13	3.99675	33.947368	7.1572	5.173695
14	4.532942	35.338412	6.629076	4.814206
15	5.084036	36.719102	6.190776	4.501501
16	5.648648	38.096776	5.821339	4.226997
17	6.225596	39.468688	5.505798	3.984089
18	6.813864	40.83479	5.233185	3.767616
19	7.412568	42.195121	4.995302	3.573478
20	8.020936	43.549771	4.785894	3.398387
21	8.638288	44.898867	4.600118	3.234766
22	9.26402	46.242555	4.434163	3.095126
23	9.897594	47.580997	4.284992	2.962941
24	10.538527	48.914356	4.150157	2.841593
25	11.186386	50.2428	4.027663	2.7298
26	11.840777	51.566494	3.915867	2.626477
27	12.501345	52.885601	3.813406	2.530694
28	13.167764	54.20028	3.71914	2.441655
29	13.839738	55.510682	3.632105	2.358673
30	14.516993	56.816955	3.551484	2.281149

CONFIDENCE LEVEL:.99
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	$\alpha 1/\alpha 2$	REDUCTION
1	0.000039	26.771726	21835.213659	50.0067
2	0.010007	23.2646	562.39577	29.292369
3	0.071367	23.209229	135.895998	20.59725
4	0.205283	24.015632	62.051271	15.858892
5	0.407145	25.147121	37.31156	12.889358
6	0.666518	26.427053	25.978223	10.857102
7	0.973755	27.781475	19.758254	9.379528
8	1.321083	29.174751	15.922441	8.25688
9	1.702411	30.58805	13.357568	7.374924
10	2.112958	32.010778	11.538086	6.663675
11	2.548926	33.436714	10.188084	6.077885
12	3.007255	34.862098	9.150588	5.587004
13	3.485444	36.284626	8.330458	5.169663
14	3.981423	37.702882	7.667024	4.810464
15	4.493457	39.116008	7.11995	4.498033
16	5.020076	40.523509	6.661466	4.223781
17	5.560021	41.92512	6.271875	3.981108
18	6.112206	43.320736	5.936854	3.764842
19	6.675684	44.71035	5.645745	3.570896
20	7.249628	46.094024	5.390476	3.39598
21	7.833305	47.471864	5.164818	3.237421
22	8.426066	48.844005	4.963899	3.093025
23	9.02733	50.210597	4.783848	2.960972
24	9.636577	51.571805	4.621563	2.839745
25	10.25334	52.927792	4.47452	2.728063
26	10.877193	54.278725	4.340652	2.624841
27	11.50775	55.624771	4.218246	2.529151
28	12.144661	56.966089	4.105876	2.440199
29	12.787604	58.30284	4.00234	2.357296
30	13.436284	59.635173	3.90662	2.279844

CONFIDENCE LEVEL:.995
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

DF	C-1	C-2	α1/α2	REDUCTION
1	0.000002	23.192274	681.204123	49.930609
2	0.002	28.714844	1718.260911	29.294246
3	0.024246	28.26948	312.632089	20.610826
4	0.090431	28.930784	122.845385	15.869013
5	0.208936	30.013072	67.206436	12.893018
6	0.378071	31.28984	43.813189	10.85574
7	0.592844	32.666149	31.733908	9.374988
8	0.847842	34.096008	24.620698	8.250542
9	1.138143	35.554968	20.034321	7.367673
10	1.459519	37.029082	16.875773	6.65607
11	1.808409	38.510084	14.589371	6.070253
12	2.181818	39.992869	12.868666	5.579543
13	2.577224	41.474264	11.532837	5.162476
14	2.992489	42.952236	10.469143	4.80361
15	3.425787	44.425505	9.604141	4.491531
16	3.875549	45.893284	8.888152	4.217637
17	4.340417	47.355082	8.286478	3.975311
18	4.819207	48.810654	7.774253	3.759381
19	5.310878	50.259878	7.333233	3.565754
20	5.814515	51.702733	6.949727	3.391139
21	6.329304	53.13928	6.613297	3.23285
22	6.854518	54.569606	6.315875	3.08872
23	7.389507	55.993844	6.051087	2.956907
24	7.933686	57.412138	5.813879	2.835904
25	8.486524	58.824638	5.600169	2.724432
26	9.047542	60.231511	5.406637	2.621402
27	9.616302	61.632923	5.230567	2.525893
28	10.192403	63.029037	5.069686	2.437108
29	10.77548	64.420015	4.922108	2.35436
30	11.365196	65.806021	4.786241	2.277051

CONFIDENCE LEVEL:.999
SHORTEST CONFIDENCE INTERVAL FOR THE STANDARD DEVIATION.

APPENDIX C

ILLUSTRATIONS

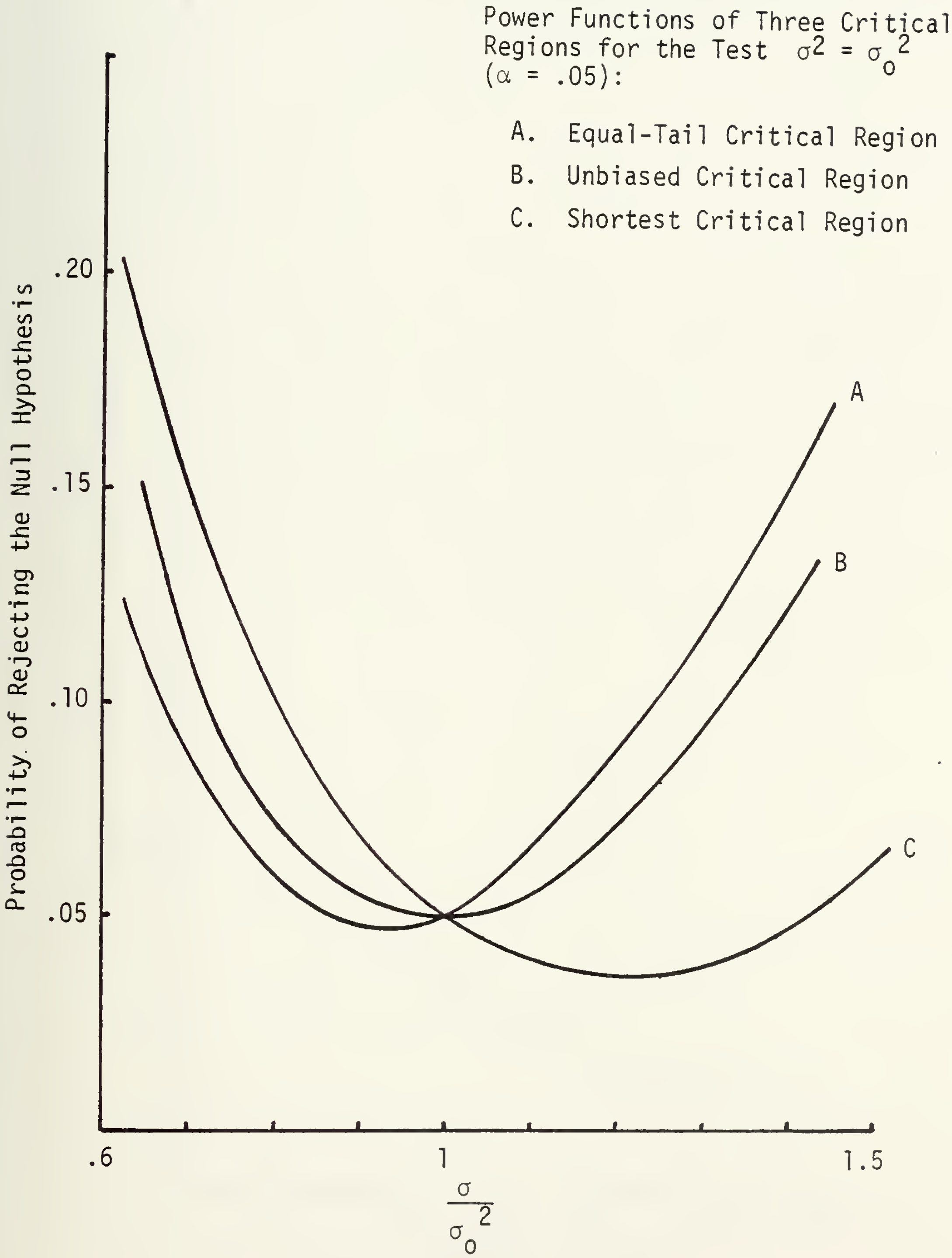


FIGURE 1

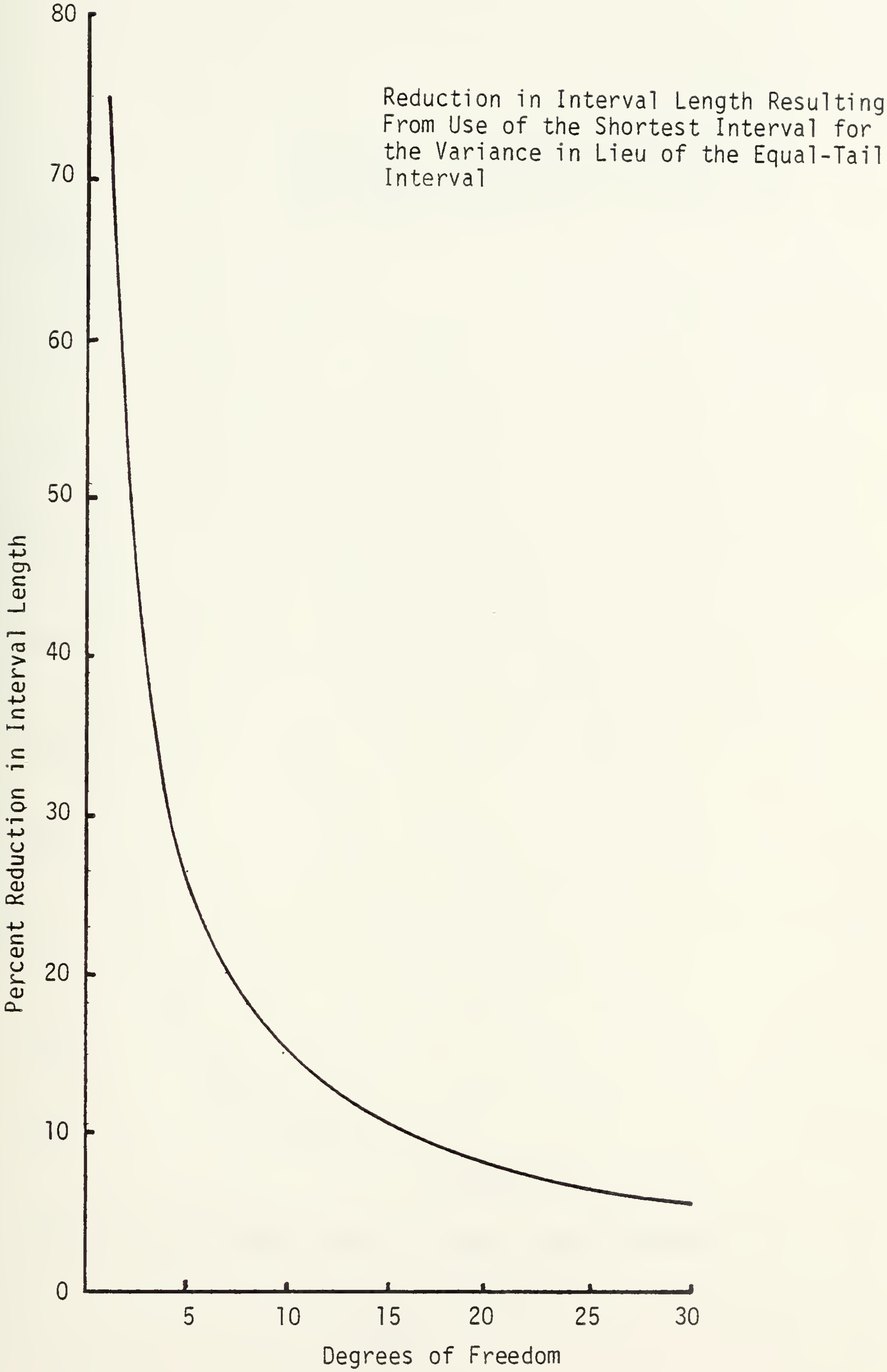

Reduction in Interval Length Resulting From Use of the Shortest Interval for the Variance in Lieu of the Equal-Tail Interval
80
70
60
50
40
30
20
10
0
Percent Reduction in Interval Length
5
10
15
20
25
30
Degrees of Freedom

FIGURE 2

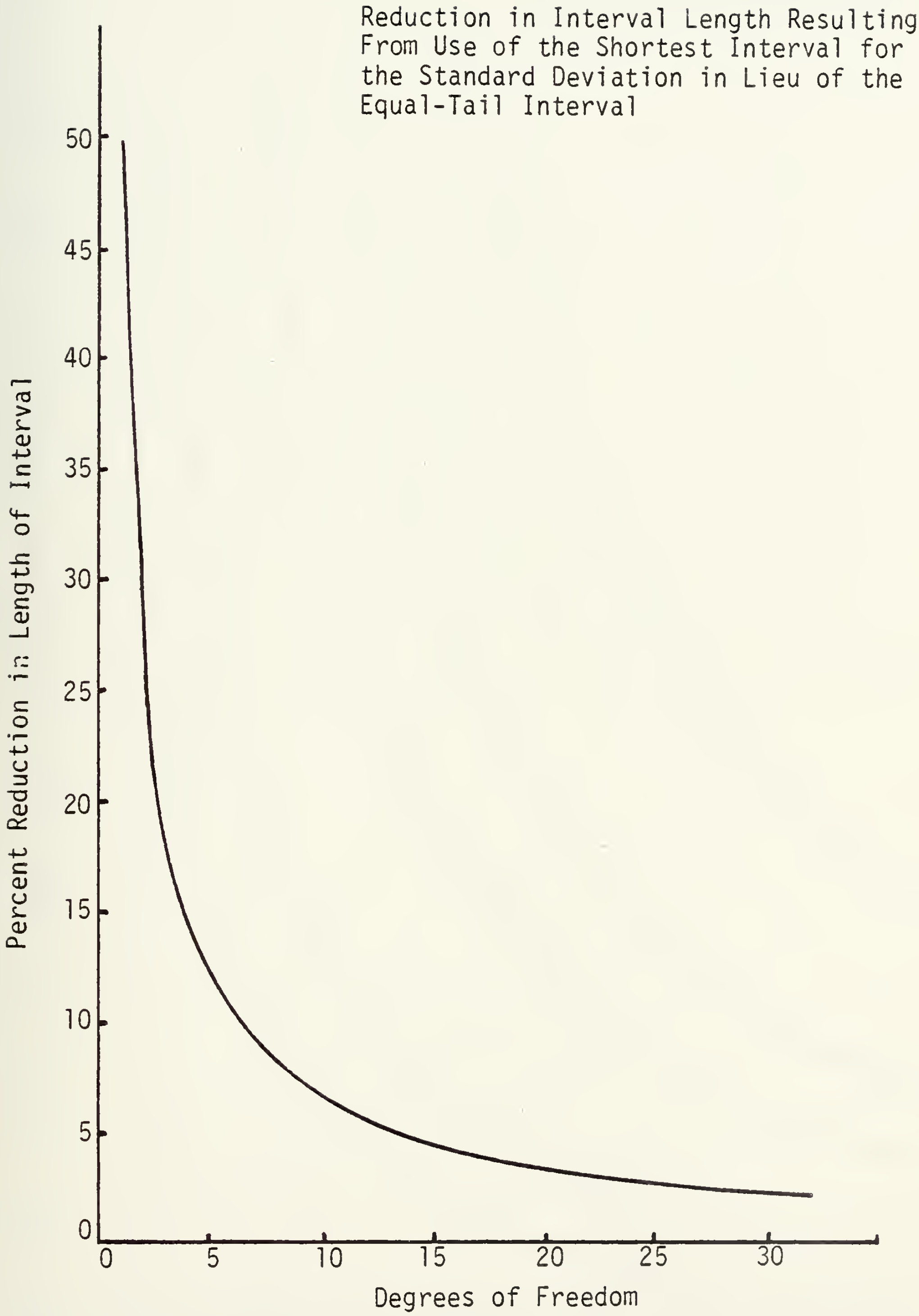

Reduction in Interval Length Resulting From Use of the Shortest Interval for the Standard Deviation in Lieu of the Equal-Tail Interval
Percent Reduction in Length of Interval
0
5
10
15
20
25
30
35
40
45
50
0
5
10
15
20
25
30
Degrees of Freedom

FIGURE 3

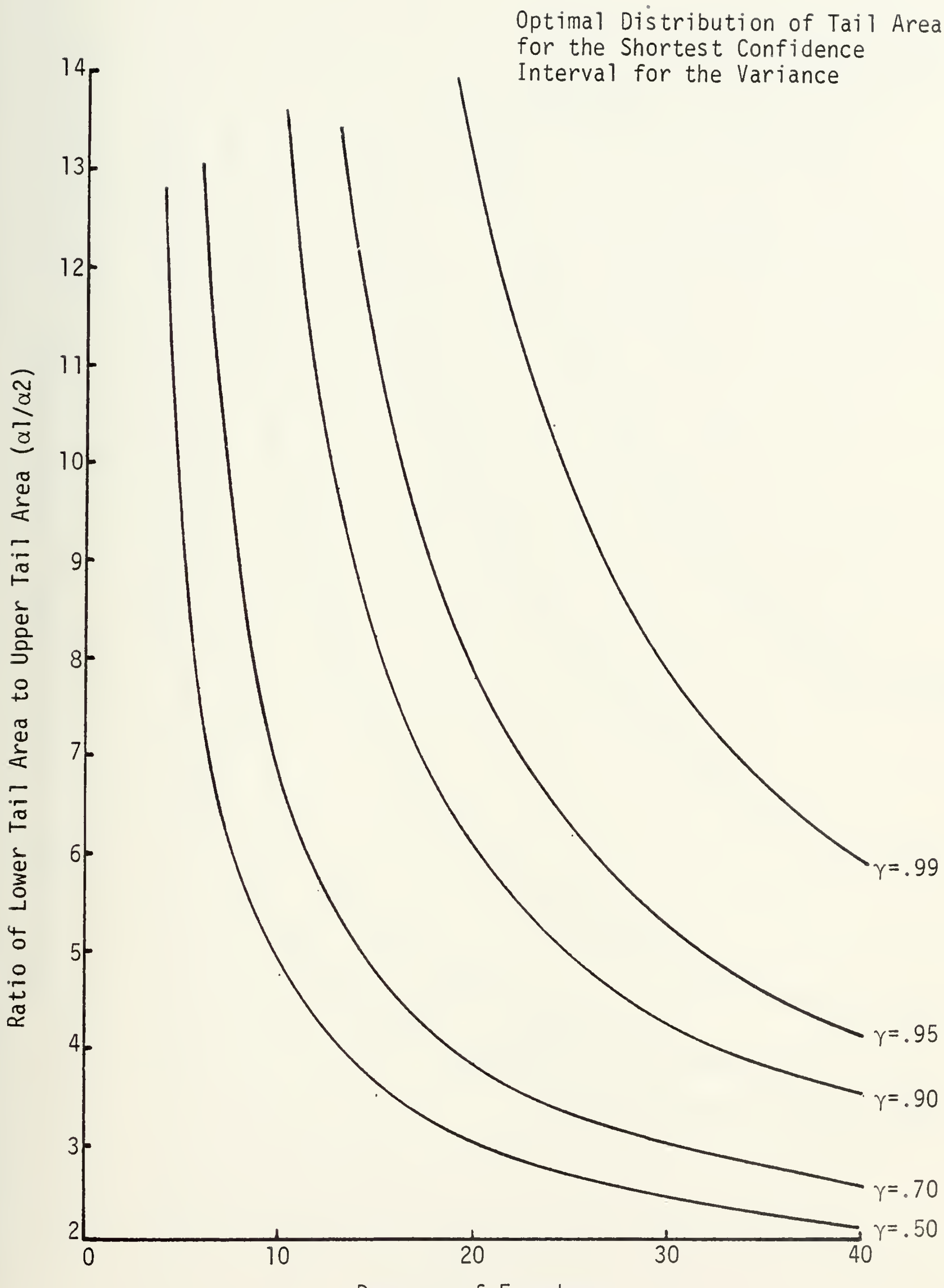
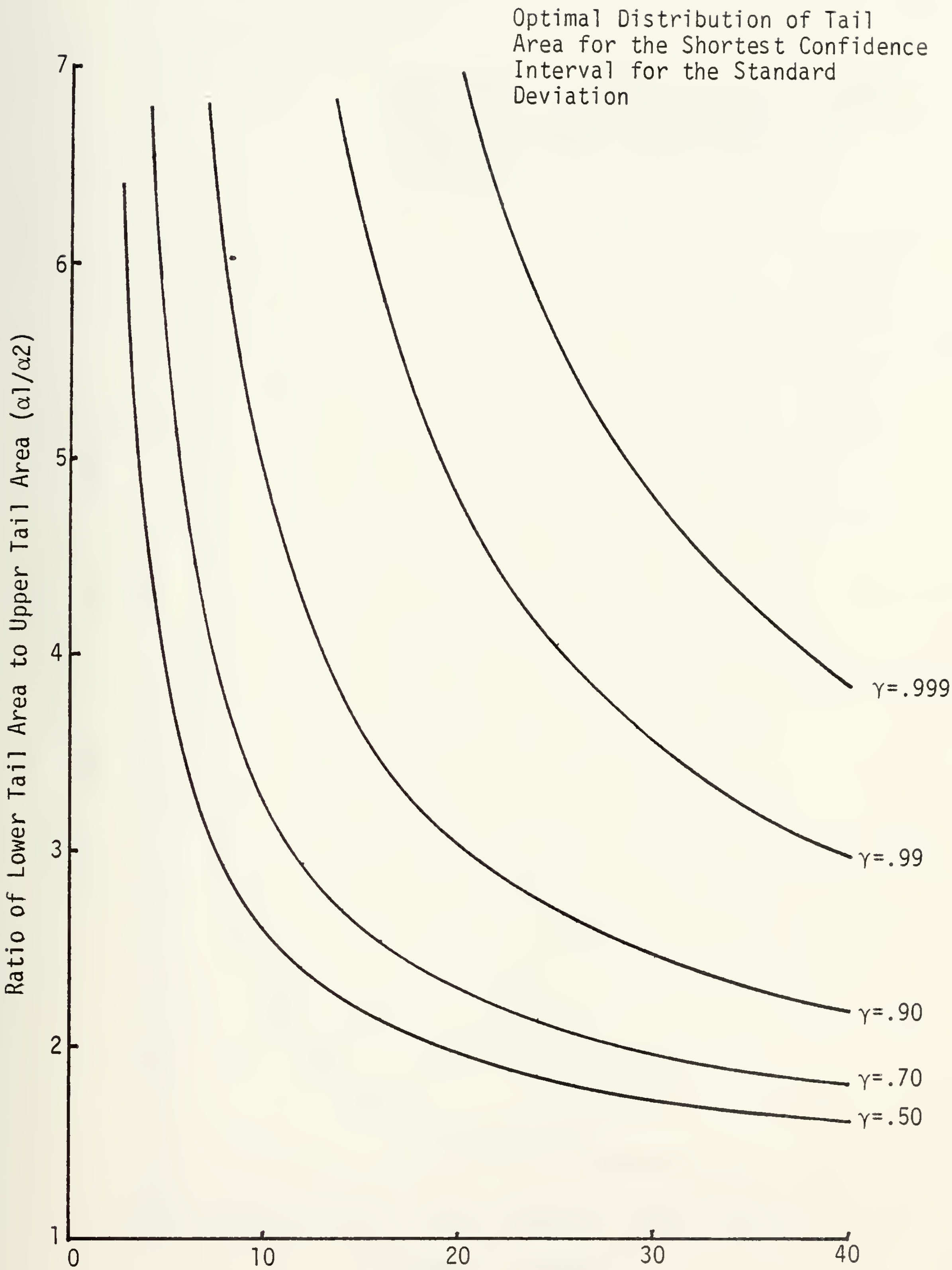
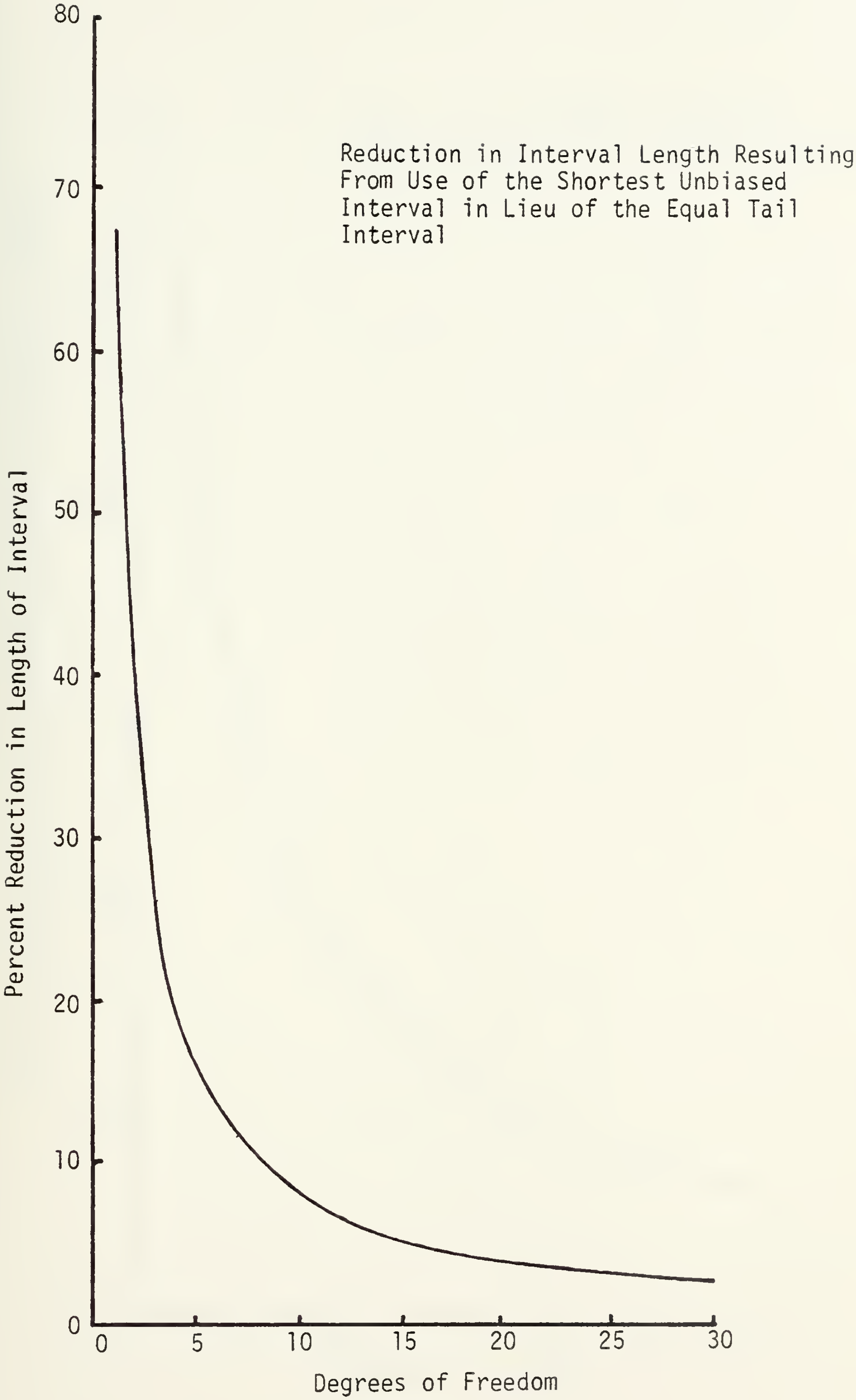

Optimal Distribution of Tail Area
for the Shortest Confidence
Interval for the Variance
Ratio of Lower Tail Area to Upper Tail Area (α1/α2)
14
13
12
11
10
9
8
7
6
5
4
3
2
0
10
20
30
40
Degrees of Freedom
γ=.99
γ=.95
γ=.90
γ=.70
γ=.50

FIGURE 4



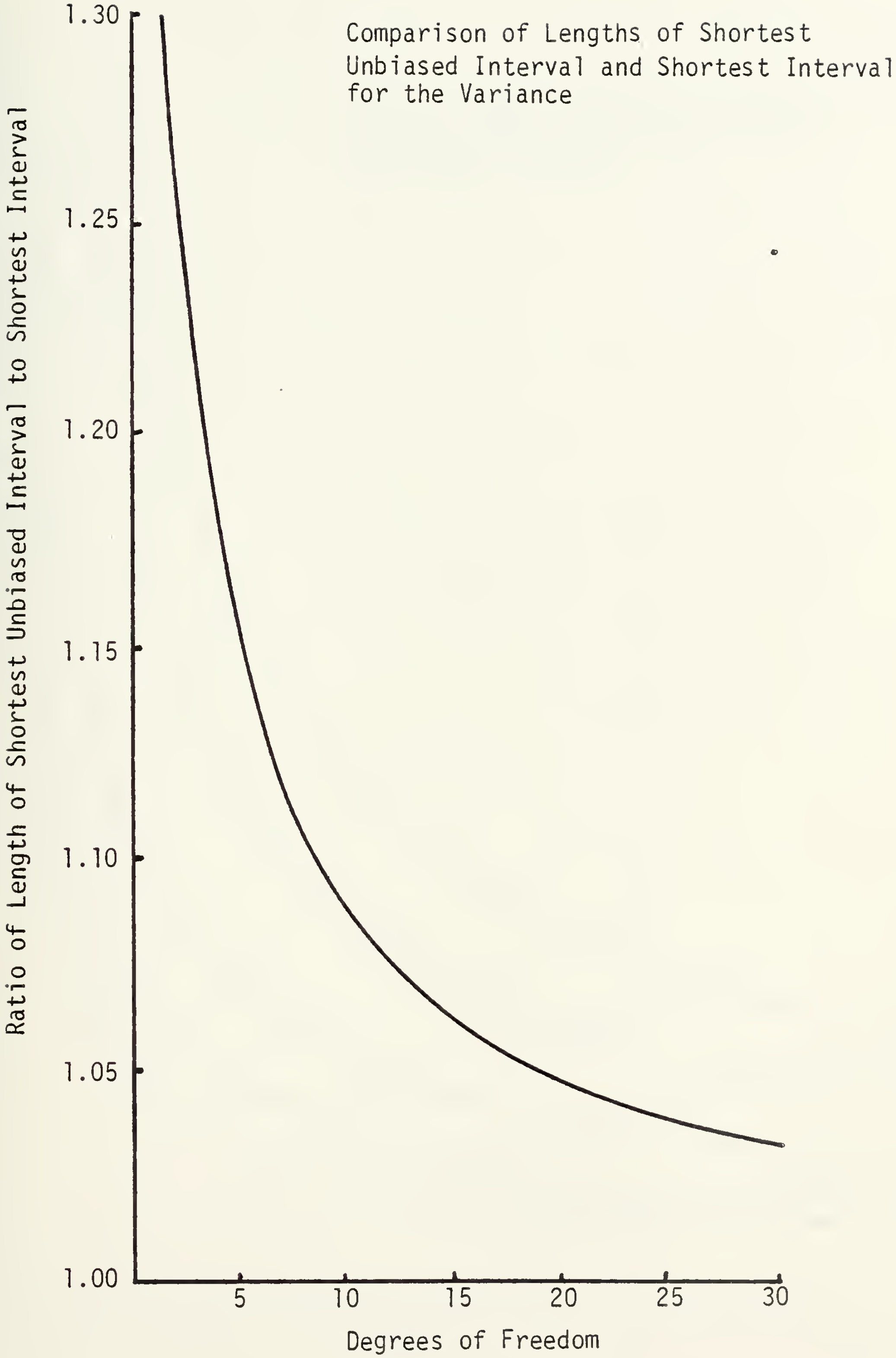
Degrees of Freedom

FIGURE 5



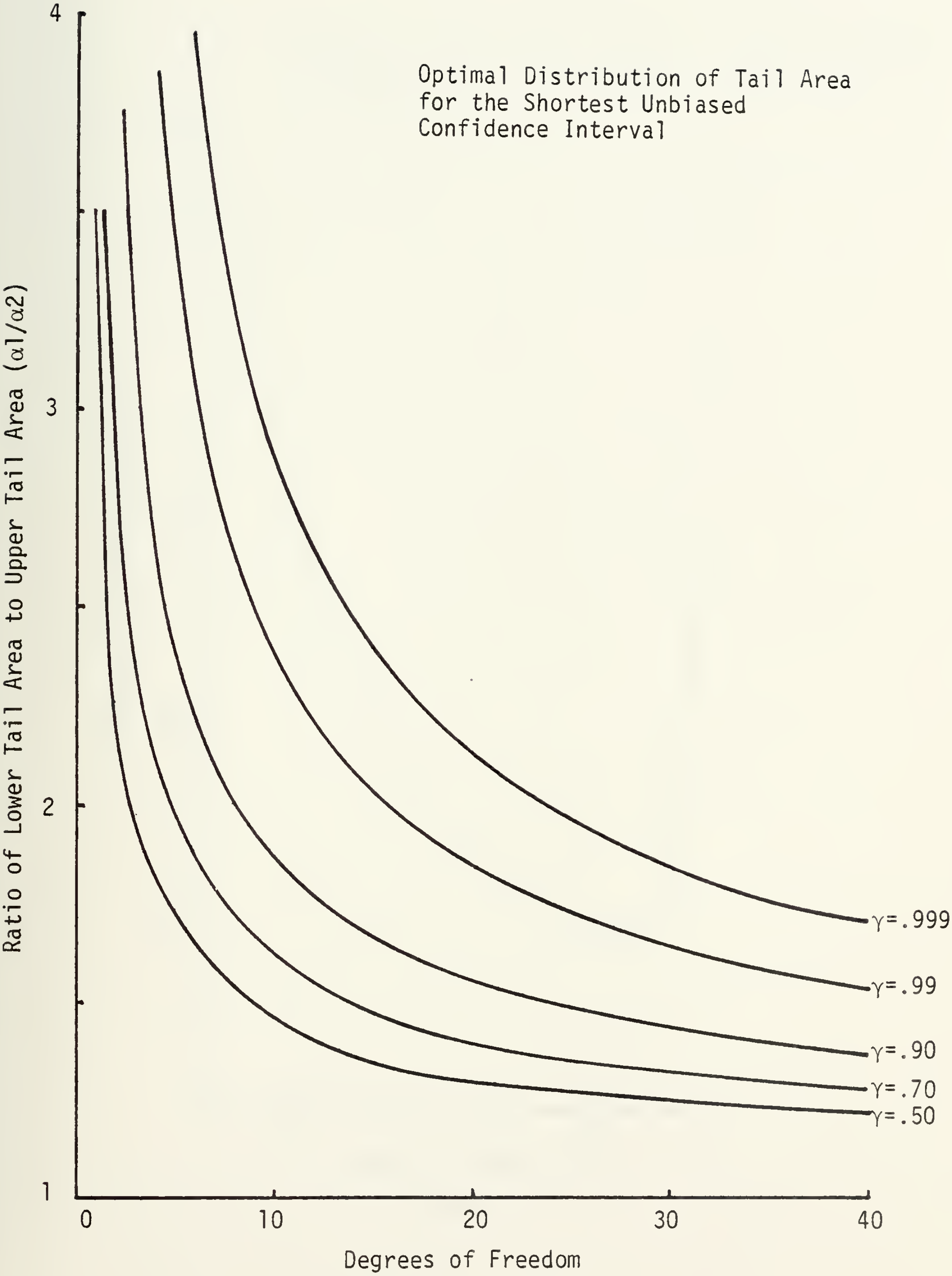
Reduction in Interval Length Resulting From Use of the Shortest Unbiased Interval in Lieu of the Equal Tail Interval
Percent Reduction in Length of Interval
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
Degrees of Freedom

FIGURE 6



Comparison of Lengths of Shortest
Unbiased Interval and Shortest Interval
for the Variance
Ratio of Length of Shortest Unbiased Interval to Shortest Interval
1.30
1.25
1.20
1.15
1.10
1.05
1.00
5
10
15
20
25
30
Degrees of Freedom

FIGURE 7



Optimal Distribution of Tail Area
for the Shortest Unbiased
Confidence Interval
Ratio of Lower Tail Area to Upper Tail Area (α1/α2)
4
3
2
1
γ=.999
γ=.99
γ=.90
γ=.70
γ=.50
0
10
20
30
40
Degrees of Freedom

FIGURE 8

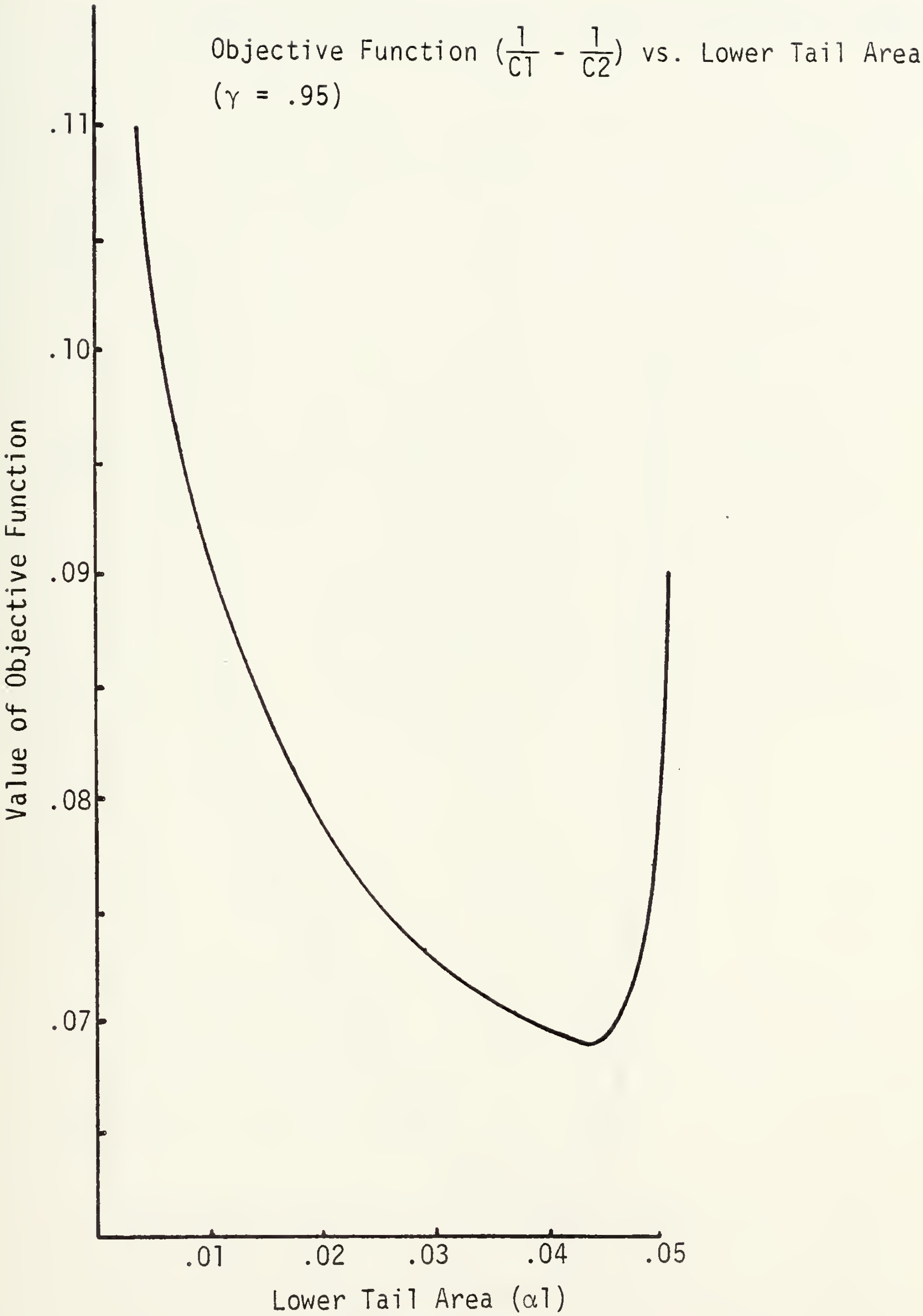

Objective Function $(\frac{1}{C1} - \frac{1}{C2})$ vs. Lower Tail Area
$(\gamma = .95)$
.11
.10
.09
.08
.07
Value of Objective Function
.01
.02
.03
.04
.05
Lower Tail Area ($\alpha 1$)

FIGURE 9


Objective Function $(\frac{1}{\sqrt{C1}} - \frac{1}{\sqrt{C2}})$ vs. Lower Tail Area
(γ = .95)
.19
.18
.17
.16
.15
Value of Objective Function
.01
.02
.03
.04
.05
Lower Tail Area (α1)

FIGURE 10

Objective Function ($\frac{C2}{C1}$) vs. Lower Tail Area
(γ = .95)
Value of Objective Function
.44
.42
.40
.38
.36
.01
.02
.03
.04
.05
Lower Tail Area (α1)

FIGURE 11

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Thesis
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c.1 Improved confidence intervals for the variance and standard deviation of a normal population.